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# Serre's reduction of linear partial differential systems with holonomic adjoints


Thomas Cluzeau* and Alban Quadrat†


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**Abstract:** Given a linear functional system (e.g., ordinary/partial differential system, differential time-delay system, difference system), Serre's reduction aims at finding an equivalent linear functional system which contains fewer equations and fewer unknowns. The purpose of this paper is to study Serre's reduction of underdetermined linear systems of partial differential equations with either polynomial, formal power series or analytic coefficients and with holonomic adjoints in the sense of algebraic analysis. We prove that these linear partial differential systems can be defined by means of a single linear partial differential equation. In the case of polynomial coefficients, we give an algorithm to compute the corresponding equation.



**Key-words:** Serre's reduction, underdetermined linear systems of partial differential equations, holonomic $D$-modules, constructive module theory, mathematical systems theory, symbolic computation.



* University of Limoges; CNRS; XLIM UMR 6172, DMI, 123 avenue Albert Thomas, 87060 Limoges Cedex, France, `cluzeau@ensil.unilim.fr`.

† INRIA Saclay - Ile-de-France, DISCO project, CNRS-SUPELEC, 3 rue Joliot Curie, 91192 Gif-sur-Yvette Cedex, France, `alban.quadrat@inria.fr`



Centre de recherche INRIA Saclay – Île-de-France
Parc Orsay Université
4, rue Jacques Monod, 91893 ORSAY Cedex
Téléphone : +33 1 72 92 59 00

# Réduction de Serre des systèmes linéaires d'équations aux dérivées partielles dont les adjoints sont holonomes

**Résumé :** Etant donné un système fonctionnel linéaire (e.g., système d'équations différentielles ordinaires, système d'équations aux dérivées partielles, système d'équations différentielles à retard, système d'équations aux différences), la réduction de Serre a pour but de trouver un système fonctionnel linéaire équivalent contenant moins d'équations et d'inconnues. L'objectif de ce papier est l'étude de la réduction de Serre des systèmes linéaires sous-déterminés d'équations aux dérivées partielles à coefficients polynomiaux, séries formelles ou séries localement convergentes, dont les adjoints sont holonomes au sens de l'analyse algébrique. Nous prouvons que de tels systèmes peuvent être définis par une seule équation aux dérivées partielles. Dans le cas des coefficients polynomiaux, nous donnons un algorithme permettant de calculer l'équation correspondante.

**Mots-clés :** Réduction de Serre, systèmes linéaires sous-déterminés d'équations aux dérivées partielles, $D$-modules holonomes, théorie constructive des modules, théorie mathématique des systèmes, calcul formel.

# 1 Introduction

One of the main goals of symbolic computation is the problem of rewriting linear/polynomial/algebraic/differential systems of equations in such a way that interesting information on the systems, which does not clearly appear in their original forms, can be easily extracted from their new forms (e.g., Gaussian elimination, Smith or Jacobson normal forms, Gröbner or Janet bases, triangular sets, formal integrability).

*Serre's reduction problem* aims at simplifying linear functional systems (e.g., ordinary (OD) or partial differential (PD) equations, time-delay equations, difference equations) in the sense of finding an equivalent presentation of the linear functional system which contains fewer unknowns and fewer equations. Serre's reduction generally helps studying the structural properties of the linear functional system and it can sometimes be used to compute its closed-form solutions. This problem also finds applications in numerical analysis. In module theory, Serre's reduction is related to the problem of characterizing the minimal number of generators (and relations) of a module finitely presented by a full row rank matrix. The efficient generation of ideals of commutative polynomial rings and its interpretation in terms of complete intersection of affine algebraic varieties of codimension 2 were the reasons for which Serre studied this problem ([30]).

The constructive study of Serre's reduction problem has recently been initiated in [2, 3]. Despite a precise mathematical characterization of the existence of Serre's reduction (see Section 4), to effectively recognize it and compute its reduced form is quite an issue. Moreover, as indicated by Serre in [30], this problem is connected to the difficult problem of recognizing whether or not certain projective or stably free modules are free (e.g., Serre's conjecture for the commutative polynomial case, nowadays known as the Quillen-Suslin theorem (see, e.g., [29])). An important case where Serre's reduction can be constructively tested is the case of a full row rank matrix $R \in D^{q\times p}$, where $D = k[x_1, \ldots, x_n]$ and $k$ is a computable field, for which the $D$-module $D^q/(R\, D^p)$ is *0-dimensional*, namely, is a finite-dimensional $k$-vector space. In this case, we can test whether or not two invertible square matrices $V$ and $W$ exist such that $V\, R\, W = \operatorname{diag}(I_r, S)$, where $\operatorname{diag}(A, B)$ stands for the block-diagonal matrix formed by the two matrices $A$ and $B$, and $I_r$ is the $r \times r$ identity matrix. In particular, the computation of the matrices $V$ and $W$ requires the computation of bases of finitely generated free $D$-modules using an implementation of the Quillen-Suslin theorem such as can be found in the QuillenSuslin package for instance ([14]). As explained in [3], the class of controllable linear OD time-delay systems studied in control theory fits in the above situation, which explained why it was possible in [2, 3, 9, 26] to exhibit Serre's reductions for many linear OD time-delay systems studied in the literature of control theory.

The purpose of this paper is to study Serre's reduction of underdetermined linear PD systems. More precisely, we focus on the situation which generalizes the case of a 0-dimensional $D = k[x_1, \ldots, x_n]$-module $D^q/(R\, D^p)$, namely, the case where the right $D = A_n(k)$-module $D^q/(R\, D^p)$ is *holonomic* in the sense of algebraic analysis (see [1, 20] and the references therein), where $A_n(k)$ denotes the *first (polynomial) Weyl algebra*, i.e., the noncommutative ring of PD operators with polynomial coefficients over a base field $k$ of characteristic 0. In this case, if $p - q \geq 1$, then, combining a classical result in algebraic analysis which asserts that a holonomic module is cyclic (see Section 3) with Stafford's theorem proving that finitely generated projective left $A_n(k)$-modules are free when their ranks are at least 2 (see Section 2), we prove in Section 5 that the left $D$-module $M = D^{1\times p}/(D^{1\times q}\, R)$ always admits Serre's reduction, i.e., there exists always a row vector $Q \in D^{1\times(p-q+1)}$ such that $M \cong D^{1\times(p-q+1)}/(D\, Q)$. If $\mathcal{F}$ is a left $D$-module (e.g., $\mathcal{F} = \mathbb{R}[x_1, \ldots, x_n]$, $\mathbb{R}(x_1, \ldots, x_n)$, $C^\infty(\mathbb{R}^n)$, $\mathcal{D}'(\mathbb{R}^n)$), then this result shows that the solution space $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$ of the linear PD system defined by the matrix $R \in D^{q\times p}$ of PD operators is equivalent to the solution space defined by a sole linear PD equation $\ker_{\mathcal{F}}(Q.) = \{\zeta \in \mathcal{F}^{(p-q+1)} \mid Q\,\zeta = 0\}$. In particular, the knowledge of $\ker_{\mathcal{F}}(Q.)$ fully characterizes $\ker_{\mathcal{F}}(R.)$ and conversely. In order to compute the matrix $Q$, we first need to compute a cyclic element for the cyclic right $D$-module $D^q/(R D^p)$ using, for instance, an algorithm developed in [19], and then compute bases of certain finitely generated free left $D$-modules using, for instance, the algorithm obtained in [27] and implemented in the Stafford package ([27]). Moreover, if $q \geq 3$, then we prove that the matrix $R$ is equivalent to $\mathrm{diag}(I_{q-1}, Q)$, i.e., two square invertible matrices $V$ and $W$ exist such that $V\,R\,W = \mathrm{diag}(I_{q-1}, Q)$. The corresponding algorithms, described in Section 5, are implemented in the Serre package ([9]) built upon the OreModules package ([6]).

Extensions of the above results are then studied when polynomial coefficients are replaced by formal power series or locally convergent power series coefficients. Using the recent extension of Stafford's theorem to the case of the ring $D$ of OD operators with either formal power series or locally convergent series coefficients ([28]), we prove that every linear OD equations with either power series or locally convergent power series coefficients and represented by a full row rank matrix $R \in D^{q\times p}$, where $p-q \geq 1$, can be defined by a row vector $Q \in D^{1\times(p-q+1)}$, which yields $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(Q.)$ for all left $D$-modules $\mathcal{F}$. Moreover, if $q \geq 3$, then the matrix $R$ is equivalent to the matrix $\mathrm{diag}(I_{q-1}, Q)$. This result is particularly interesting in control theory where $p-q$ is the number of inputs of the system. We point out that the above results allow us to avoid singularities which may appear in the Jacobson normal forms of the matrix $R$ (see [18, 32]). Finally, in Section 4, we explain the connections between Serre's reduction of first order linear OD system or linear evolution PD systems with the concepts of observability and controllability developed in control theory and show how Serre's reduction extends the concept of *cyclic vectors* classically used in the literature of linear OD systems with coefficients in a differential field (e.g., $\mathbb{R}(t)$, $\mathbb{R}[\![t]\!][t^{-1}]$, $\mathbb{R}\{t\}[t^{-1}]$) (see, e.g., [4, 10]) to linear OD systems with coefficients in a differential ring (e.g., $\mathbb{R}[t]$, $\mathbb{R}[\![t]\!]$, $\mathbb{R}\{t\}$).

This paper is an extension of the results obtained in [8].

## 2 Algebraic analysis approach to linear systems theory

In this section, we recall the algebraic analysis approach to mathematical systems theory that will be used in the next sections. Moreover, we introduce the main notations and state a few results which will be used in what follows.

We shall denote by $D^{1\times p}$ (resp., $D^q$) the left (resp., right) $D$-module formed by row (resp., column) vectors of length $p$ (resp., $q$) with entries in $D$ and by $R \in D^{q\times p}$ a $q \times p$ matrix with entries in $D$. The *general linear group* of $D$ of index $p$, namely,

$$\mathrm{GL}_p(D) = \{U \in D^{p\times p} \mid \exists\, V \in D^{p\times p} :\; U\,V = V\,U = I_p\},$$

is the subgroup of the ring $D^{p\times p}$ formed by invertible (*unimodular*) matrices.

In what follows, we shall use the following notations:

$$\begin{array}{rcl} .R : D^{1\times q} & \longrightarrow & D^{1\times p}, \\ \mu & \longmapsto & \mu\,R, \end{array} \qquad \begin{array}{rcl} R. : D^{p} & \longrightarrow & D^{q} \\ \eta & \longmapsto & R\,\eta. \end{array}$$

Within algebraic analysis (see, e.g., [1, 5, 20]), a linear functional system (e.g., linear systems of ODEs or PDEs, OD time-delay equations, difference equations) can be studied by means of module theory and homological algebra. More precisely, if $D$ is a noncommutative polynomial ring of functional operators (e.g., OD or PD operators, time-delay operators, shift operators, difference operators), $R \in D^{q\times p}$ and $\mathcal{F}$ a left $D$-module, then the *linear functional system* $\ker_{\mathcal{F}}(R.) \triangleq \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$, i.e., the abelian group formed by the $\mathcal{F}$-solutions of the linear system $R\,\eta = 0$, can be studied by means of the left $D$-module $M \triangleq D^{1\times p}/(D^{1\times q}\,R)$ *finitely presented* by the matrix $R$. Indeed, Malgrange's remark ([21]) asserts the existence of the abelian group isomorphism

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}), \tag{1}$$

where $\hom_D(M, \mathcal{F})$ denotes the abelian group (i.e., $\mathbb{Z}$-module) of left $D$-*homomorphisms* from $M$ to $\mathcal{F}$ (i.e., maps $f : M \longrightarrow \mathcal{F}$ satisfying $f(d_1\,m_1 + d_2\,m_2) = d_1\,f(m_1) + d_2\,f(m_2)$ for all $d_1,\, d_2 \in D$ and all $m_1,\, m_2 \in M$) and $\cong$ is the equivalence relation of being *isomorphic* ([29]).

Let us describe the isomorphism (1). To do that, we first give an explicit description of $M$ in terms of *generators and relations*. Let $\pi : D^{1\times p} \longrightarrow M = D^{1\times p}/(D^{1\times q}\,R)$ be the canonical projection onto $M$, namely, the left $D$-homomorphism which sends a row vector of $D^{1\times p}$ to its residue class $\pi(\lambda)$ in $M$, $\{f_j\}_{j=1,\ldots,p}$ the *standard basis* of $D^{1\times p}$, namely, $f_j$ is the row vector of length $p$ defined by 1 at the $j^{\text{th}}$ entry and 0 elsewhere, and $y_j = \pi(f_j)$ the residue class of $f_j$ in $M$ for $j = 1, \ldots, p$. Since every element $m \in M$ is the residue class of an element $\lambda = (\lambda_1 \ \ldots \ \lambda_p) \in D^{1\times p}$, then we get

$$m = \pi(\lambda) = \pi\left(\sum_{j=1}^{p} \lambda_j\, f_j\right) = \sum_{j=1}^{p} \lambda_j\, \pi(f_j) = \sum_{j=1}^{p} \lambda_j\, y_j,$$

which shows that $\{y_j\}_{j=1,\ldots,p}$ is a family of generators of the left $D$-module $M$. Now, if $R_{i\bullet}$ denotes the $i^{\text{th}}$ row of $R$, then $R_{i\bullet} \in D^{1\times q}\,R$, which yields $\pi(R_{i\bullet}) = 0$ and thus

$$\forall\, i = 1, \ldots, q, \quad \pi(R_{i\bullet}) = \pi\left(\sum_{j=1}^{p} R_{ij}\, f_j\right) = \sum_{j=1}^{p} R_{ij}\, \pi(f_j) = \sum_{j=1}^{p} R_{ij}\, y_j = 0, \tag{2}$$

which shows that the set of generators $\{y_j\}_{j=1,\ldots,p}$ of $M$ satisfies the left $D$-linear relations (2) and their left $D$-linear combinations. Therefore, if we set $y = (y_1 \ \ldots \ y_p)^T \in M^p$, then (2) becomes $R\,y = 0$.

Let $\chi : \ker_{\mathcal{F}}(R.) \longrightarrow \hom_D(M, \mathcal{F})$ be the $\mathbb{Z}$-homomorphism defined by $\chi(\eta) = \phi_\eta$, where $\phi_\eta(\pi(\lambda)) = \lambda\,\eta \in \mathcal{F}$ for all $\lambda \in D^{1\times p}$. The $\mathbb{Z}$-homomorphism $\phi_\eta$ is well-defined since $\pi(\lambda) = \pi(\lambda')$ yields $\pi(\lambda - \lambda') = 0$, i.e., $\lambda - \lambda' = \mu\,R$ for a certain $\mu \in D^{1\times q}$, and thus $\phi_\eta(\pi(\lambda)) = \lambda\,\eta = \lambda'\,\eta + \mu\,R\,\eta = \lambda'\,\eta = \phi_\eta(\pi(\lambda'))$. Moreover, $\chi$ is injective since $\phi_\eta = 0$ yields $\lambda\,\eta = 0$ for all $\lambda \in D^{1\times p}$, and thus $\eta_j = f_j\,\eta = 0$ for all $j = 1, \ldots, p$, i.e., $\eta = 0$. It is also surjective since for all $\phi \in \hom_D(M, \mathcal{F})$, $\eta = (\phi(y_1) \ \ldots \ \phi(y_p))^T \in \mathcal{F}^p$ satisfies $\chi(\eta) = \phi$ and

$$\forall\, i = 1, \ldots, q, \quad \sum_{j=1}^{p} R_{ij}\, \eta_j = \sum_{j=1}^{p} R_{ij}\, \phi(y_j) = \phi\left(\sum_{j=1}^{p} R_{ij}\, y_j\right) = \phi(0) = 0,$$

i.e., $\eta \in \ker_{\mathcal{F}}(R.)$. Thus, the $\mathbb{Z}$-homomorphism $\chi$ is an isomorphism.

**Theorem 1** ([21]). *Let $D$ be a ring, $M = D^{1\times p}/(D^{1\times q}\,R)$ the left $D$-module finitely presented by the matrix $R \in D^{q\times p}$, $\pi : D^{1\times p} \longrightarrow M$ the canonical projection onto $M$, $\{f_j\}_{j=1,\ldots,p}$ the* *standard basis of* $D^{1\times p}$, $y_j = \pi(f_j)$ *for* $j = 1, \ldots, p$, *and* $\mathcal{F}$ *a left* $D$*-module. Then, we have the following* $\mathbb{Z}$*-isomorphism:*

$$\begin{array}{rcl} \hom_D(M, \mathcal{F}) & \longrightarrow & \ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\} \\ \phi & \longmapsto & \eta = (\phi(y_1) \ \ldots \ \phi(y_p))^T. \end{array} \tag{3}$$

*Hence, there is a one-to-one correspondence between the elements of* $\hom_D(M, \mathcal{F})$ *and the elements of* $\ker_{\mathcal{F}}(R.)$.

**Remark 1.** Theorem 1 shows that $\ker_{\mathcal{F}}(R.)$ can be studied by means of the finitely presented left $D$-module $M$ and the left $D$-module $\mathcal{F}$: $M = D^{1\times p}/(D^{1\times q}\,R)$ intrinsically defines the linear system of equations defined by the matrix $R \in D^{q\times p}$ and $\mathcal{F}$ is the functional space where we seek the solutions of the linear functional system.

In what follows, $D$ will denote a *noncommutative noetherian domain*, namely, a unital ring satisfying that $d\,d'$ is not necessarily equal to $d'\,d$ for all $d$, $d' \in D$, containing no nontrivial zero-divisors, i.e., $d\,d' = 0$ yields $d = 0$ or $d' = 0$, and every left (resp., right) ideal of $D$ is finitely generated, i.e., can be generated by a finite family of elements of $D$ as a left (resp., right) $D$-module (see, e.g., [29]).

A *differential ring* $(A, \{\delta_1, \ldots, \delta_n\})$ is a commutative ring $A$ equipped with $n$ *commuting derivations* $\delta_i : A \longrightarrow A$ for $i = 1, \ldots, n$, namely, maps satisfying:

$$\forall\, i, j = 1, \ldots, n, \quad \forall\, a_1,\, a_2 \in A, \quad \left\{ \begin{array}{l} \delta_i \circ \delta_j = \delta_j \circ \delta_i, \\ \delta_i(a_1 + a_2) = \delta_i(a_1) + \delta_i(a_2), \\ \delta_i(a_1\,a_2) = \delta_i(a_1)\,a_2 + a_1\,\delta_i(a_2). \end{array} \right.$$

If we take $a_1 = a_2 = 1$, then the above equality yields $\delta_i(1) = 2\,\delta_i(1)$, i.e., $\delta_i(1) = 0$. If $A$ is a field and $a \in A \setminus \{0\}$, then $\delta_i(a)\,a^{-1} + a\,\delta_i(a^{-1}) = \delta_i(a\,a^{-1}) = \delta_i(1) = 0$, which yields $\delta_i(a^{-1}) = -a^{-2}\,\delta_i(a)$ and $A$ is then called a *differential field*.

In what follows, we shall mainly focus on the differential ring $\left(A, \left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}\right)$, where $A = k[x_1, \ldots, x_n]$, $k[\![x_1, \ldots, x_n]\!]$ (i.e., the ring of formal power series at 0 with coefficients in $k$), where $k$ is a field of characteristic 0 (e.g., $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$), $k\{x_1, \ldots, x_n\}$ where $k = \mathbb{R}$ or $\mathbb{C}$ (i.e., the ring of locally convergent power series at 0 or equivalently, the ring of germs of real analytic or holomorphic functions at 0) or on the differential fields $A = k$ and $k(x_1, \ldots, x_n)$, where $k$ is a field.

The *ring* $D = A\langle\partial_1, \ldots, \partial_n\rangle$ *of PD operators* in $\partial_1, \ldots, \partial_n$ with coefficients in the differential ring $(A, \{\delta_1, \ldots, \delta_n\})$ is the noncommutative polynomial ring in the $\partial_i$'s with coefficients in the ring $A$ satisfying:

$$\forall\, i, j = 1, \ldots, n, \quad \forall\, a \in A, \quad \partial_i\,\partial_j = \partial_j\,\partial_i, \quad \partial_i\,a = a\,\partial_i + \delta_i(a).$$

An element $d \in D$ can be written as $d = \sum_{0\le|\nu|\le r} a_\nu\,\partial^\nu$, where $\nu = (\nu_1 \ \ldots \ \nu_n)^T \in \mathbb{N}^n$, $|\nu| = \nu_1 + \cdots + \nu_n$, $\partial^\nu = \partial_1^{\nu_1} \ldots \partial_n^{\nu_n}$ and $a_\nu \in A$. If $n = 1$, then we shall simply use the notations $\delta = \frac{d}{dt}$ instead of $\delta_1$, $\partial$ instead of $\partial_1$ and $k[t]$, $k(t)$, $k[\![t]\!]$ and $k\{t\}$ instead of $k[x_1]$, $k(x_1)$, $k[\![x_1]\!]$ and $k\{x_1\}$.

The *first* (polynomial) and the *second* (rational) *Weyl algebra* are defined by:

$$A_n(k) \triangleq k[x_1, \ldots, x_n]\langle\partial_1, \ldots, \partial_n\rangle, \quad B_n(k) \triangleq k(x_1, \ldots, x_n)\langle\partial_1, \ldots, \partial_n\rangle.$$

The ring $D = A\langle\partial_1, \ldots, \partial_n\rangle$, where $A = k$, $k[x_1, \ldots, x_n]$, $k(x_1, \ldots, x_n)$ or $k[\![x_1, \ldots, x_n]\!]$, and $k$ is a field, or $k\{x_1, \ldots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, is a noetherian domain (see, e.g., [23]).

Let us recall a few definitions of module theory.

**Definition 1** ([17, 29])**.** Let $D$ be a left noetherian domain and $M$ a *finitely generated* left $D$-module, namely, $M$ can be generated by a finite family of elements of $M$ as a left $D$-module.

1. $M$ is *free* if there exists $r \in \mathbb{N} = \{0, 1, \ldots\}$ such that $M \cong D^{1\times r}$. Then, $r$ is called the *rank* of the free left $D$-module $M$ and is denoted by $\operatorname{rank}_D(M)$.

2. $M$ is *stably free* if there exist $r$, $s \in \mathbb{N}$ such that $M \oplus D^{1\times s} \cong D^{1\times r}$. Then, $r - s$ is called the *rank* of the stably free left $D$-module $M$.

3. $M$ is *projective* if there exist $r \in \mathbb{N}$ and a left $D$-module $N$ such that $M \oplus N \cong D^{1\times r}$, where $\oplus$ denotes the direct sum of left $D$-modules.

4. $M$ is *torsion-free* if the *torsion left $D$-submodule of $M$*, namely,
$$t(M) = \{m \in M \mid \exists\, d \in D \setminus \{0\} :\ d\, m = 0\},$$
is reduced to 0, i.e., if $t(M) = 0$. The elements of $t(M)$ are called the *torsion elements*.

5. $M$ is *torsion* if $t(M) = M$, i.e., if every element of $M$ is a torsion element.

6. $M$ is *cyclic* if there exists $m \in M$ such that $M = D\, m \triangleq \{d\, m \mid d \in D\}$.

A free module is clearly stably free (take $s = 0$ in 2 of Definition 1), a stably free module is projective (take $P = D^{1\times s}$ in 3 of Definition 1) and a projective module is torsion-free (since it can be embedded into a free, and thus, into a torsion-free module).

The converses of the previous results are generally not true. However, some of them hold in the following interesting situations for mathematical systems theory ([17, 23, 28, 29]).

**Theorem 2.** *1. If $D$ is a* principal left ideal domain*, namely, every left ideal of the domain $D$ is principal (e.g., the ring $A\langle\partial\rangle$ of OD operators with coefficients in a differential field $A$ such as $A = k$, $k(t)$ and $k[\![t]\!][t^{-1}]$, where $k$ is a field of characteristic 0 (e.g., $k = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$), or $k\{t\}[t^{-1}]$, where $k = \mathbb{R}$ or $\mathbb{C}$), then every finitely generated torsion-free left $D$-module is free.*

*2. If $D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring with coefficients in a field $k$, then every finitely generated projective $D$-module is free (*Quillen-Suslin theorem*).*

*3. If $D$ is the Weyl algebra $A_n(k)$ or $B_n(k)$, where $k$ is a field of characteristic 0, then every finitely generated projective left $D$-module is stably free and every finitely generated stably free left $D$-module of rank at least 2 is free (Stafford's theorem).*

*4. If $D = A\langle\partial\rangle$ is the ring of OD operators with coefficients in a differential ring $A = k[\![t]\!]$, where $k$ is a field of characteristic 0, or $k\{t\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, then every finitely generated projective left $D$-module is stably free and every finitely generated stably free left $D$-module of rank at least 2 is free.*

If the matrix $R$ has *full row rank*, namely, $\ker_D(.R) \triangleq \{\mu \in D^{1\times q} \mid \mu\, R = 0\} = 0$, i.e., the rows of $R$ are left $D$-linearly independent, then the next proposition characterizes when the left $D$-module $M = D^{1\times p}/(D^{1\times q}\, R)$ is a stably free or free module.

**Theorem 3** (see, e.g., [14, 27])**.** *Let $D$ be a noetherian domain, $R \in D^{q\times p}$ a full row rank matrix, i.e., $\ker_D(.R) = 0$, and the finitely presented left $D$-module $M = D^{1\times p}/(D^{1\times q}\, R)$.*

1. *$M$ is a projective left $D$-module iff $M$ is a stably free left $D$-module.*

2. *$M$ is a stably free left $D$-module of rank $p-q$ iff $R$ admits a* right inverse*s, namely, iff there exists a matrix $S \in D^{p\times q}$ satisfying $R\,S = I_q$.*

3. *$M$ is a free left $D$-module of rank $p-q$ iff there exists $U \in \mathrm{GL}_p(D)$ such that:*

$$R\,U = (I_q \quad 0).$$

*If $U = (S \quad Q)$, where $S \in D^{p\times q}$ and $Q \in D^{p\times(p-q)}$, then we have*

$$\begin{array}{rcl} \psi : M & \longrightarrow & D^{1\times(p-q)} \\ \pi(\lambda) & \longmapsto & \lambda\, Q, \end{array} \qquad \begin{array}{rcl} \psi^{-1} : D^{1\times(p-q)} & \longrightarrow & M \\ \mu & \longmapsto & \pi(\mu\, T), \end{array}$$

*where the matrix $T \in D^{(p-q)\times p}$ is defined by:*

$$U^{-1} = \begin{pmatrix} R \\ T \end{pmatrix} \in D^{p\times p}.$$

*In particular, $M \cong D^{1\times p}\, Q = D^{1\times(p-q)}$. The matrix $Q$ is then called an* injective parametrization *of $M$. Finally, $\{\pi(T_{i\bullet})\}_{i=1,\ldots,p-q}$ is a basis of the free left $D$-module $M$ of rank $p-q$.*

The Quillen-Suslin theorem (resp., the Stafford's theorem) has recently been implemented in the QuillenSuslin package ([14]) (resp., Stafford package ([27])). Hence, we can compute bases and injective parametrizations of finitely generated free left $D$-modules, where $D = k[x_1, \ldots, x_n]$ and $k = \mathbb{Q}$ or $\mathbb{F}_p$ ($p$ is a prime number), $A_n(\mathbb{Q})$ or $B_n(\mathbb{Q})$.

# 3 Holonomic $D$-modules

In this section, we consider the ring $D = A\langle\partial_1, \ldots, \partial_n\rangle$ of PD operators with coefficients in the differential ring $A = k$, $k[x_1, \ldots, x_n]$, $k(x_1, \ldots, x_n)$ or $k[\![x_1, \ldots, x_n]\!]$, where $k$ is a field of characteristic 0, or $k\{x_1, \ldots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$. The purpose of this section is to recall two important results on the so-called *holonomic $D$-modules*, namely, the forthcoming Theorems 4 and 5, which will play a central role in Section 5.

The ring $D$ is endowed with the *order filtration* defined by:

$$\forall\, r \in \mathbb{N}, \quad D_r = \left\{ \sum_{0\leq|\alpha|\leq r} a_\alpha\, \partial^\alpha \mid a_\alpha \in A \right\}.$$

We can check that the filtration properties hold, namely:

1. $\forall\, r,\, s \in \mathbb{N}$, $r \leq s \;\Rightarrow\; D_r \subseteq D_s$.

2. $D = \bigcup_{r\in\mathbb{N}} D_r$.

3. $\forall\, r,\, s \in \mathbb{N}$: $D_r\, D_s \subseteq D_{r+s}$.

The ring $D$ is called a *filtered ring* and an element of $D_r$ has a *degree* less than or equal to $r$. We can easily check that $D_0 = A$ and $D_r$ is a finitely generated $A$-module.

If $d_1,\, d_2 \in D$, then we can define the *bracket* of $d_1$ and $d_2$ by $[d_1, d_2] \triangleq d_1\, d_2 - d_2\, d_1$. If $d_1 \in D_r$ and $d_2 \in D_s$, then $d_1\, d_2$ and $d_2\, d_1$ belong to $D_{r+s}$ since $D_r\, D_s \subseteq D_{r+s}$ and $D_s\, D_r \subseteq D_{r+s}$, and we can check that $[d_1, d_2] \in D_{r+s-1}$, i.e., $[D_r, D_s] \subseteq D_{r+s-1}$.

Let us now introduce the following $A$-module

$$\mathrm{gr}(D) = \bigoplus_{r \in \mathbb{N}} D_r / D_{r-1},$$

where we have set $D_{-1} = 0$. If $\pi_r : D_r \longrightarrow D_r/D_{r-1}$ is the canonical projection for all $r \in \mathbb{N}$, then the $A$-module $\mathrm{gr}(D)$ inherits a ring structure defined by

$$\forall\, d_1 \in D_r, \quad \forall\, d_2 \in D_s, \quad \left\{ \begin{array}{l} \pi_r(d_1) + \pi_s(d_2) \triangleq \pi_t(d_1 + d_2) \in D_t/D_{t-1}, \\ \pi_r(d_1)\, \pi_s(d_2) \triangleq \pi_{r+s}(d_1\, d_2) \in D_{r+s}/D_{r+s-1}, \end{array} \right.$$

where $t = \max(r, s)$. The ring $\mathrm{gr}(D)$ is called the *graded ring* associated with the order filtration of $D$.

Let $\chi_i \triangleq \pi_1(\partial_i) \in D_1/D_0$ for all $i = 1, \ldots, n$. Then, $\pi_1([\partial_i, \partial_j]) = 0$ and $\pi_1([\partial_i, a]) = 0$ for all $a \in A$ and all $i, j = 1, \ldots, n$ since $[\partial_i, \partial_j] = 0$ and $[\partial_i, a] \in D_0$, which shows that

$$\mathrm{gr}(D) = A[\chi_1, \ldots, \chi_n]$$

is the commutative polynomial ring in $\chi_1, \ldots, \chi_n$ with coefficients in the commutative noetherian ring $A$.

We can now extend the concepts of filtered and graded rings to modules.

**Definition 2** ([1, 11, 20])**.** Let $M$ be a finitely generated left $D = A\langle \partial_1, \ldots, \partial_n \rangle$-module.

1. A *filtration* of $M$ is a sequence $\{M_q\}_{q \in \mathbb{N}}$ of $A$-submodules of $M$ (with the convention that $M_{-1} = 0$) such that:

   (a) $\forall\, q,\, r \in \mathbb{N},\, q \leq r \;\Rightarrow\; M_q \subseteq M_r$.

   (b) $M = \bigcup_{q \in \mathbb{N}} M_q$.

   (c) $\forall\, q,\, r \in \mathbb{N}$: $D_r\, M_q \subseteq M_{q+r}$.

   The left $D$-module $M$ is then called a *filtered module*

2. The associated *graded* $\mathrm{gr}(D)$-module $\mathrm{gr}(M)$ is defined by:

   (a) $\mathrm{gr}(M) = \bigoplus_{q \in \mathbb{N}} M_q/M_{q-1}$.

   (b) For every $d \in D_r$ and every $m \in M_q$, we set

   $$\pi_r(d)\, \sigma_q(m) \triangleq \sigma_{q+r}(d\, m) \in M_{q+r}/M_{q+r-1},$$

   where $\sigma_q : M_q \longrightarrow M_q/M_{q-1}$ is the canonical projection for all $q \in \mathbb{N}$.

3. A filtration $\{M_q\}_{q \in \mathbb{N}}$ is called a *good filtration* if one of the two following equivalent conditions is satisfied:

(a) $M_q$ is a finitely generated $A$-module for all $q \in \mathbb{N}$ and there exists $p \in \mathbb{N}$ such that $D_r\, M_p = M_{p+r}$ for all $r \in \mathbb{N}$.

(b) $\mathrm{gr}(M) = \bigoplus_{q \in \mathbb{N}} M_q / M_{q-1}$ is a finitely generated $\mathrm{gr}(D)$-module.

**Example 1.** Let $M$ be a finitely generated left $D$-module defined by a family of generators $\{y_1, \ldots, y_p\}$. Then, the filtration $M_q = \sum_{i=1}^{p} D_q\, y_i$ is a good filtration of $M$ since

$$\mathrm{gr}(M) = \sum_{i=1}^{p} \mathrm{gr}(D)\, y_i,$$

which proves that $\mathrm{gr}(M)$ is a finitely generated left $\mathrm{gr}(D)$-module.

If $M$ is a finitely generated left $D = A\langle \partial_1, \ldots, \partial_n \rangle$-module, then $\mathrm{gr}(M)$ is a finitely generated module over the commutative polynomial ring $\mathrm{gr}(D) = A[\chi_1, \ldots, \chi_n]$. Hence, we are back to the realm of commutative algebra. Based on techniques of algebraic geometry and commutative algebra, one can then use invariants of $\mathrm{gr}(M)$ (e.g., dimension, multiplicity) to characterize the left $D$-module $M$ (see, e.g., [1, 11, 20]).

**Definition 3** ([13])**.** A *proper prime ideal* of a commutative ring $A$ is an ideal $\mathfrak{p} \subsetneq A$ which satisfies that $a\, b \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Endowed with the *Zariski topology* defined by the Zariski-closed sets $V(I) = \{\mathfrak{p} \in \mathrm{spec}(A) \mid I \subseteq \mathfrak{p}\}$, where $I$ is an ideal of $A$, the set of all the proper prime ideals of $A$, denoted by $\mathrm{spec}(A)$, is a topological space.

We can now introduce the concept of a *characteristic variety* of a differential module.

**Proposition 1** ([1, 11, 20])**.** *Let $M$ be a finitely generated left $D = A\langle \partial_1, \ldots, \partial_n \rangle$-module and $G = \mathrm{gr}(M)$ the associated graded $\mathrm{gr}(D) = A[\chi_1, \ldots, \chi_n]$-module for a good filtration of $M$. Then, the* characteristic ideal $I(M)$ *of $M$ is the ideal of the commutative polynomial ring $\mathrm{gr}(D)$ defined by:*

$$I(M) = \sqrt{\mathrm{ann}(G)} \triangleq \{a \in \mathrm{gr}(D) \mid \exists\, n \in \mathbb{N} :\; a^n\, G = 0\}.$$

*It does not depend on the good filtration of $M$. The* characteristic variety *of $M$ is then the subset of $\mathrm{spec}(\mathrm{gr}(D))$ defined by:*

$$\mathrm{char}_D(M) = \{\mathfrak{p} \in \mathrm{spec}(\mathrm{gr}(D)) \mid \sqrt{\mathrm{ann}(G)} \subseteq \mathfrak{p}\}.$$

According to Example 1, every finitely generated left $D = A\langle \partial_1, \ldots, \partial_n \rangle$-module $M$ admits a good filtration and thus a characteristic variety.

The *dimension* of the left $D$-module $M$ can be defined as the geometric dimension of the characteristic variety $\mathrm{char}_D(M)$ of $M$.

**Definition 4** ([1, 11, 20])**.** Let $M$ be a finitely generated left $D = A\langle \partial_1, \ldots, \partial_n \rangle$-module. Then, the *dimension* of $M$ is the supremum of the lengths of the chains $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \ldots \subset \mathfrak{p}_d$ of distinct proper prime ideals in the commutative ring $\mathrm{gr}(D)/I(M) = A[\chi_1, \ldots, \chi_n]/I(M)$. If $M = 0$, then we set $\dim_D(M) = -1$.

In what follows, we shall simply write $\dim(D)$ instead of $\dim_D(D)$.

**Example 2** ([1, 11, 20])**.** We have

$$\dim(k[x_1, \ldots, x_n]) = n, \quad \dim(B_n(k')) = n, \quad \dim(A\langle \partial_1, \ldots, \partial_n \rangle) = 2\, n,$$

where $k$ is a field, $k'$ is a field of characteristic 0 and $A = k[x_1, \ldots, x_n]$, $k[\![x_1, \ldots, x_n]\!]$, where $k$ is a field of characteristic 0, or $k\{x_1, \ldots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$.

**Example 3.** Let us consider the linear PD system:

$$\begin{cases} \Phi_1 = (\partial_4 - x_3\,\partial_2 - 1)\,y = 0, \\ \Phi_2 = (\partial_3 - x_4\,\partial_1)\,y = 0. \end{cases} \tag{4}$$

We can check that (4) is not *formally integrable* since

$$(\partial_4 - x_3\,\partial_2 - 1)\,\Phi_2 + (x_4\,\partial_1 - \partial_3)\,\Phi_1 = (\partial_2 - \partial_1)\,y = 0$$

is a new non-trivial first order PD equation which does not appear in (4). Adding this new equation to (4), then we can check that the new linear PD system defined by

$$\begin{cases} (\partial_4 - x_3\,\partial_2 - 1)\,y = 0, \\ (\partial_3 - x_4\,\partial_1)\,y = 0, \\ (\partial_2 - \partial_1)\,y = 0, \end{cases} \tag{5}$$

is formally integrable and *involutive* (see, e.g., [25] and the references therein). Therefore, using the Cartan-Kähler-Janet's theorem (see, e.g., [25]), there exists a formal power series (locally convergent power series) solution of (5) in a neighbourhood of the point $a = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4$ which satisfies an appropriate set of initial conditions which can be determined (see, e.g., [25]).

Using (5), the characteristic variety of the left $D = A_4(\mathbb{C})$-module $M = D/(D^{1\times 2}\,R)$ finitely presented by $R = (\partial_4 - x_3\,\partial_2 - 1 \quad \partial_3 - x_4\,\partial_1)^T$ is defined by the ideal

$$I(M) = (\chi_4 - x_3\,\chi_2,\ \chi_3 - x_4\,\chi_1,\ \chi_2 - \chi_1)$$

of $\mathrm{gr}(D) = \mathbb{C}[x_1, x_2, x_3, x_4, \chi_1, \chi_2, \chi_3, \chi_4]$. The characteristic variety $\mathrm{char}_D(M)$ of $M$ is then the affine algebraic variety of $\mathbb{C}^8$ defined by the ideal $I(M)$ of $\mathrm{gr}(D)$:

$$\mathrm{char}_D(M) = \{(x_1, x_2, x_3, x_4, \chi_1, \chi_1, x_4\,\chi_1, x_3\,\chi_1) \mid \chi_1,\ x_i \in \mathbb{C},\ i = 1, \ldots, 4\}.$$

The Krull dimension of $\mathrm{char}_D(M)$ is then 5, i.e., $\dim_D(M) = 5$.

**Definition 5** ([1, 11, 20]). Let $M$ be a non-zero finitely generated left $D = A\langle\partial_1, \ldots, \partial_n\rangle$-module. If $\dim_D(M) = n$ then $M$ is called a *holonomic* left $D$-module.

**Example 4.** The time-varying OD equation $t\,\dot{y} - y = 0$ defines the holonomic left $D = A_1(\mathbb{C})$-module $M = D/D\,(t\,\partial - 1)$. Indeed, the characteristic variety $\mathrm{char}_D(M)$ of $M$ is defined by the characteristic ideal $I(M) = (t\,\chi)$ of the commutative polynomial ring $\mathrm{gr}(D) = \mathbb{C}[t, \chi]$, which implies that $\mathrm{char}_D(M) = \{(t,\,0) \,|\, t \in \mathbb{C}\} \cup \{(0,\,\chi) \,|\, \chi \in \mathbb{C}\}$ is a 1-dimensional affine algebraic variety of $\mathbb{C}^2$, and thus $\dim_D(M) = 1$.

The next two theorems will play important roles in Section 5.

**Theorem 4** ([1, 11, 20]). *If $D = A\langle\partial\rangle$ is the ring of OD operators with coefficients in $A = k[t]$, $k[\![t]\!]$, where $k$ is a field of characteristic 0, or $k\{t\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, then a left (resp., right) $D$-module $M$ is holonomic iff $M$ is a torsion left (resp., right) $D$-module.*

**Theorem 5** ([1, 11, 20]). *If $A = k[x_1, \ldots, x_n]$, $k[\![x_1, \ldots, x_n]\!]$, where $k$ is a field of characteristic 0, or $k\{x_1, \ldots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, then a holonomic left $D = A\langle\partial_1, \ldots, \partial_n\rangle$-module $M$ is cyclic, i.e., $M$ can be generated by one element as a left $D$-module. More precisely, if $\{y_j\}_{j=1,\ldots,p}$ is a set of generators of the holonomic left $D$-module $M$, then there exist $d_2, \ldots, d_p \in D$ such that $M$ is generated by $z = y_1 + d_2\,y_2 + \cdots + d_p\,y_p$. Similar results hold for holonomic right $D$-modules.*

**Remark 2.** For $D = A_n(k)$, where $k$ is a computable field of characteristic 0 (e.g., $k = \mathbb{Q}$), a constructive algorithm for the computation of a cyclic element of a finitely presented holonomic left $D$-module $M$ is given in [19]. The corresponding algorithm is implemented in the package Serre ([9]) built upon OreModules ([6]).

# 4 Serre's reduction of linear systems

In this section, we study when a finitely presented left (resp., right) $D$-module $M = D^{1\times p}/(D^{1\times q}\, R)$ (resp., $N = D^q/(R\, D^p)$) can be generated by less than $p$ (resp., $q$) generators. Then, we recall recent results on *Serre's reduction* ([30]) obtained in [2, 3]. They will be used in Section 5 to study Serre's reduction of the class of linear PD systems with holonomic *Auslander transposes* or *adjoints*.

If $R \in D^{q\times p}$ and $\{f_j\}_{j=1,\dots,p}$ is the standard basis of $D^{1\times p}$, then the beginning of Section 2 shows that $\{y_j = \pi(f_j)\}_{j=1,\dots,p}$ is a family of generators of the finitely presented left $D$-module $M = D^{1\times p}/(D^{1\times q}\, R)$, where $\pi : D^{1\times p} \longrightarrow M$ is the canonical projection. Moreover, $\{y_j\}_{j=1,\dots,p}$ satisfies the relations $R\, y = 0$, where $y = (y_1 \ \dots \ y_p)^T$.

Let us first investigate when the left $D$-module $M = D^{1\times p}/(D^{1\times q}\, R)$ can be generated by less generators than $p$. Let $0 \le r \le p-1$ and $\Lambda \in D^{(p-r)\times p}$ be such that

$$P \triangleq \begin{pmatrix} R \\ \Lambda \end{pmatrix} \in D^{(q+p-r)\times p}$$

admits a left inverse $S = (S_1 \quad S_2) \in D^{p\times(q+p-r)}$, where $S_1 \in D^{p\times q}$ and $S_2 \in D^{p\times(p-r)}$. We note that this result always holds if we take $\Lambda = I_p$, which shows that the interesting case starts with $r \ge 1$. By hypothesis, we have $D^{1\times(q+p-r)}\, P = D^{1\times p}$, which yields

$$M = D^{1\times p}/(D^{1\times q}\, R) = (D^{1\times(q+p-r)}\, P)/(D^{1\times q}\, R), \tag{6}$$

and shows that $\{z_k = \pi(\Lambda_{k\bullet}) = \sum_{j=1}^{p} \Lambda_{kj}\, y_j\}_{k=1,\dots,p-r}$ is a family of generators of $M$. Let us give another way to understand this result. The identity $S_1\, R + S_2\, \Lambda = I_p$ yields

$$y = S_1\,(R\, y) + S_2\,(\Lambda\, y) = S_2\,(\Lambda\, y),$$

which shows that $z \triangleq \Lambda\, y \in M^{(p-r)}$ satisfies $y = S_2\, z$, i.e., $\{z_k = \sum_{j=1}^{p} \Lambda_{kj}\, y_j\}_{k=1,\dots,p-r}$ is a family of generators of $M$. In particular, if $r = p-1$, then $\Lambda \in D^{1\times p}$ and $M$ is generated by $\pi(\Lambda)$, i.e., $M = D\, \pi(\Lambda)$ is a cyclic left $D$-module.

Moreover, let $Q = (Q_1 \quad Q_2) \in D^{s\times(q+p-r)}$, where $Q_1 \in D^{s\times q}$ and $Q_2 \in D^{s\times(p-r)}$, be a matrix such that $\ker_D(.P) = D^{1\times s}\, Q$. Using the identity $R = (I_q \quad 0)\, P$, Lemma 3.1 of [7] then yields

$$\begin{aligned} M = (D^{1\times(q+p-r)}\, P)/(D^{1\times q}\, R) &\cong D^{1\times(q+p-r)}/\left(D^{1\times(q+s)}\begin{pmatrix} I_q & 0 \\ Q_1 & Q_2 \end{pmatrix}\right) \\ &\cong L \triangleq D^{1\times(p-r)}/(D^{1\times s}\, Q_2), \end{aligned}$$

where the left $D$-isomorphism $\phi : M \longrightarrow L$ is defined by $\phi(\pi(\mu\, P)) = \sigma(\mu_2)$ for all $\mu = (\mu_1 \quad \mu_2)$, where $\mu_1 \in D^{1\times q}$, $\mu_2 \in D^{1\times(p-r)}$ and $\sigma : D^{1\times(p-r)} \longrightarrow L$ is the canonical projection, and $\phi^{-1} : L \longrightarrow M$ is defined by $\phi^{-1}(\sigma(\mu_2)) = \pi(\mu_2\, \Lambda)$ for all $\mu_2 \in D^{1\times(p-r)}$. If $\{g_k\}_{k=1,\dots,p-r}$ is the standard basis of $D^{1\times(p-r)}$, then the generators $\{v_k = \sigma(g_k)\}_{k=1,\dots,p-r}$ of the left $D$-module $L$

satisfy the relations $Q_2\, v = 0$, where $v = (v_1 \; \ldots \; v_{p-r})^T$. We also have $\phi^{-1}(v_k) = \pi(g_k\, \Lambda) = \pi(\Lambda_{k\bullet}) = z_k$, i.e., $\phi(z_k) = v_k$ for all $k = 1, \ldots, p-r$, which shows that, up to isomorphism, $M$ can be generated by $p-r$ elements satisfying $s$ relations. Finally, we can interpret this result by noticing that $Q_2\, z = 0$ generates the compatibility conditions of the following inhomogeneous linear system:

$$\begin{cases} R\, y = 0, \\ \Lambda\, y = z. \end{cases}$$

Conversely, let us suppose that the left $D$-module $M = D^{1\times p}/(D^{1\times q}\, R)$ can be generated by a family of $p-r$ generators $\{z_k = \pi(\Lambda_{k\bullet}) = \sum_{j=1}^{p} \Lambda_{kj}\, y_j\}_{k=1,\ldots,p-r}$ for a certain $r$ satisfying $0 \leq r \leq p-1$ and a certain matrix $\Lambda \in D^{(p-r)\times p}$. Then, there exists a matrix $U \in D^{p\times(p-r)}$ such that $y_j = \sum_{k=1}^{p-r} U_{jk}\, z_k$ for all $j = 1, \ldots, p$, which yields $(I_p - U\,\Lambda)\, y = 0$, and thus there exists a matrix $V \in D^{p\times q}$ such that $I_p = U\,\Lambda + V\, R$, which proves that the matrix $P \triangleq (R^T \quad \Lambda^T)^T \in D^{(q+p-r)\times p}$ admits a left inverse.

**Lemma 1.** *The finitely presented left $D$-module $M = D^{1\times p}/(D^{1\times q}\, R)$ can be generated by $p-r$ elements, where $r$ satisfies $0 \leq r \leq p-1$, iff there exists $\Lambda \in D^{(p-r)\times p}$ such that $P = (R^T \quad \Lambda^T)^T \in D^{(q+p-r)\times p}$ admits a left inverse. Then, $\{\pi(\Lambda_{k\bullet})\}_{k=1,\ldots,p-r}$ is a family of generators of $M$, where $\pi : D^{1\times p} \longrightarrow M$ is the standard projection onto $M$.*

Let us illustrate Lemma 1 with the class of linear time-varying first order OD systems. Let $D = A\langle\partial\rangle$ be the ring of OD operators with coefficients in a noetherian differential integral domain $\left(A, \frac{d}{dt}\right)$, $F \in A^{n\times n}$, $R = \partial\, I_n - F \in D^{n\times n}$. Using Lemma 1, the finitely presented left $D$-module $M = D^{1\times n}/(D^{1\times n}\, R)$ can be generated by $p \triangleq n-r$ elements $\{\pi(H_{k\bullet})\}_{k=1,\ldots,p}$, where $0 \leq r \leq n-1$ and $H \in D^{p\times n}$, iff $P = ((\partial\, I_n - F)^T \quad H^T)^T$ admits a left inverse, i.e., iff the left $D$-module $E \triangleq D^{1\times n}/(D^{1\times(n+p)}\, P)$ is reduced to 0. In terms of generators and relations, the left $D$-module $E$ is generated by $\{x_i\}_{i=1,\ldots,n}$ which satisfies the relations:

$$\begin{cases} \partial\, x - F\, x = 0, \\ H\, x = 0. \end{cases} \tag{7}$$

Since $\partial\, x = F\, x$ and $F \in A^{n\times n}$, without loss of generality, we can take $H \in A^{p\times n}$. Pre-multiplying the second equation of (7) by $\partial$ and taking into account the first equation of (7), we obtain $H\,\partial\, x + \dot{H}\, x = 0$ and thus $(H\, F + \dot{H})\, x = 0$. We can now repeat the same operations with this new zero-order equation and so on. We obtain:

$$(7) \quad \Leftrightarrow \quad \begin{cases} \partial\, x - F\, x = 0, \\ H_i\, x = 0, \quad \forall\, i \in \mathbb{N}, \end{cases}$$

where the matrices $H_i$'s are inductively defined by:

$$\begin{cases} H_0 = H, \\ H_{i+1} = H_i\, F + \dot{H}_i. \end{cases} \tag{8}$$

Let $\mathcal{L}_j \triangleq \sum_{i=0}^{j} A^{1\times p}\, H_i$ be the $A$-submodule of the left $A$-module $A^{1\times n}$ generated by the $H_i$'s for $i = 1, \ldots, j$. Since $\mathcal{L}_j \subseteq \mathcal{L}_{j+1}$ for all $j \in \mathbb{N}$, the sequence $(\mathcal{L}_j)_{j\in\mathbb{N}}$ of $A$-submodules of the noetherian $A$-module $A^{1\times n}$ stabilizes, namely, there exists $s \in \mathbb{N}$ such that:

$$\forall\, j \in \mathbb{N}, \quad \mathcal{L}_{s+j} = \mathcal{L}_s = \sum_{i=0}^{s} A^{1\times p}\, H_i.$$

Therefore, we get:

$$(7) \quad \Leftrightarrow \quad \begin{cases} \partial\, x - F\, x = 0, \\ H_0\, x = 0, \\ \quad \vdots \\ H_s\, x = 0. \end{cases}$$

If $\mathcal{L}_s = A^{1\times n}$, i.e., if $(H_0^T \; \ldots \; H_s^T)^T$ admits a left inverse, then (7) yields $x = 0$, i.e., $E = D^{1\times n}/(D^{1\times (n+p)}\, P) = 0$, which shows that $P$ admits a left inverse, and thus $M$ is generated by $\{\pi(H_{k\bullet})\}_{k=1,\ldots,p}$ by Lemma 1.

Conversely, let us suppose that there exists a left inverse $(X \quad Y)$ of the matrix $P$, where $X \in D^{n\times n}$ and $Y \in D^{n\times p}$, i.e., $X\,(\partial\, I_n - F) + Y\, H = I_n$. Using (8), we have:

$$\forall\, i \in \mathbb{N}, \quad \partial\, H_i = H_i\, \partial + \dot{H}_i = H_i\, \partial + H_{i+1} - H_i\, F = H_i\,(\partial\, I_n - F) + H_{i+1}.$$

Moreover, we have:

$$\begin{aligned} \forall\, i \in \mathbb{N}, \quad \partial^2\, H_i &= \partial\,(\partial\, H_i) = \partial\,(H_i\,(\partial\, I_n - F) + H_{i+1}) = (\partial\, H_i)\,(\partial\, I_n - F) + \partial\, H_{i+1} \\ &= (H_i\,(\partial\, I_n - F) + H_{i+1})\,(\partial\, I_n - F) + H_{i+1}\,(\partial\, I_n - F) + H_{i+2} \\ &= H_i\,(\partial\, I_n - F)^2 + 2\, H_{i+1}\,(\partial\, I_n - F) + H_{i+2}. \end{aligned}$$

More generally, we can inductively prove:

$$\forall\, i \in \mathbb{N}, \quad \forall\, l \in \mathbb{N}, \quad \partial^l\, H_i = \sum_{j=0}^{l} \frac{l!}{j!\,(l-j)!}\, H_{i+j}\,(\partial\, I_n - F)^{l-j}. \tag{9}$$

If $Y = \sum_{l=0}^{d} C_l\, \partial^l$, where $C_l \in A^{n\times p}$, then using $H_0 = H$ and (9) with $i = 0$, we obtain

$$\begin{aligned} \forall\, l \in \mathbb{N}, \quad Y\, H &= \sum_{l=0}^{d} C_l\, \partial^l\, H = \sum_{l=0}^{d} \sum_{j=0}^{l} \frac{l!}{j!\,(l-j)!}\, C_l\, H_j\,(\partial\, I_n - F)^{l-j} \\ &= \sum_{l=0}^{d} C_l\, H_l + Z\,(\partial\, I_n - F), \end{aligned}$$

for a certain matrix $Z$, which implies that $X\,(\partial\, I_n - F) + Y\, H = I_n$ is equivalent to:

$$(X + Z)\,(\partial\, I_n - F) + \sum_{l=0}^{d} C_l\, H_l = I_n. \tag{10}$$

Since each entry of the matrix $\partial\, I_n - F$ has degree 1 in $\partial$, the identity (10) can only hold when $X = -Z$, which yields $\sum_{l=0}^{s} C_l\, H_l = I_n$ and shows that the existence of a left inverse of $P$ implies the existence of a left inverse of the matrix $(H_0^T \; \ldots \; H_s^T)^T$.

**Corollary 1.** *Let $\left(A, \frac{d}{dt}\right)$ be a noetherian differential integral domain, $F \in A^{n\times n}$, $D = A\langle\partial\rangle$ the ring of OD operators with coefficients in $A$ and $R = \partial\, I_n - F \in D^{n\times n}$. Then, the finitely presented left $D$-module $M = D^{1\times n}/(D^{1\times n}\, R)$ can be generated by $p$ elements $\{\pi(H_{k\bullet})\}_{k=1,\ldots,p}$, where $H \in A^{p\times n}$, iff the matrix $P = (R^T \quad H^T)^T$ admits a left inverse, i.e., iff there exists $s \in \mathbb{N}$ such that the matrix $(H_0^T \; \ldots \; H_s^T)^T$ admits a left inverse, where the matrices $H_i$'s are defined by (8).*

If we want to check whether or not the left $D$-module $M$ can be generated by $\{\pi(H_{k\bullet})\}_{k=1,\dots,p}$, then we first have to saturate (7) (*formal integrability* of (7)), namely, find $s \in \mathbb{N}$ such that $\mathcal{L}_{s+j} = \mathcal{L}_s$ for all $j \in \mathbb{N}$, and then test whether or not the matrix $(H_0^T \ \dots \ H_s^T)^T$ admits a left inverse. For instance, if $A = k\{t\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, then the last step can be achieved by checking whether or not $\operatorname{rank}_{\mathbb{R}}(H_0^T \ \dots \ H_s^T)^T(0)$ is equal to $n$ (an element $a$ of the local ring $A$ is an invertible iff $a \notin (t)$, i.e., $a(0) \neq 0$).

In control theory (see, e.g., [15, 31]), the state $x$ of $\dot{x} = F\,x$ is said to be *observable* from the output $y \triangleq H\,x$ if $x$ can be expressed by means of $y$ and its derivatives, which means that the matrix $P = ((\partial\, I_n - F)^T \quad H^T)^T$ admits a left inverse. Using the above results, $x$ is observable by $y = H\,x$ iff $M = D^{1\times n}/(D^{1\times n}\,R)$ can be generated by $\{y_k = \pi(H_{k\bullet}) = \sum_{j=1}^n H_{kj}\,x_j\}_{k=1,\dots,p}$, namely, $M = \sum_{k=1}^p D\,y_k$. Moreover, in control theory, the sequence of matrices $H_i$'s is called the *observability distribution* and the condition that there exists $s \in \mathbb{N}$ such that the matrix $(H_0^T \ \dots \ H_s^T)^T$ admits a left inverse is the *observability condition*. Hence, the search for a generating set of $M$ can be interpreted as the search for outputs $y = H\,x$ of the linear system $\dot{x} = F\,x$ such that $x$ is observable. Finally, if $A = k[t]$ or $k[\![t]\!]$, where $k$ is a field of characteristic 0, or $A = k\{t\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, then Theorem 5 shows that the state $x$ of $\dot{x} = F\,x$ can always be observed by a single output $y = H\,x$, where $H \in A^{1\times n}$ is a certain matrix which can be computed by means of Algorithm 3 of [19] in the case of $A = k(t)$, where $k$ is a computable field of characteristic 0 (see Remark 2).

If $A$ is now a differential field (e.g., $A = k\{t\}[t^{-1}]$, where $k = \mathbb{R}$ or $\mathbb{C}$), then, in the literature of linear OD systems, the vector $H \in A^{1\times n}$ is called a *cyclic vector* of $\dot{x} = F\,x$ if $\det(H_0^T \ \dots \ H_{n-1}^T)^T \neq 0$, where the row vectors $H_i$'s are defined by (8). In particular, if $A = k(t)$, where $k$ is a field of characteristic 0, then a cyclic vector always exists for $\dot{x} = F\,x$, where $F \in A^{n\times n}$ ([4, 10]).

Finally, if $\frac{d}{dt}$ is a trivial derivation of $A$, i.e., $\frac{d}{dt}\,a = \dot{a} = 0$ for all $a \in A$ (e.g., $A = \mathbb{R}$), then (8) yields $H_i = H\,F^i$ for all $i \in \mathbb{N}$, and the Cayley-Hamilton theorem (see, e.g., [29]) for a commutative ring $A$ then shows that $F^n = \sum_{i=0}^{n-1} a_i\,F^i$ for certain $a_i \in A$, and thus $\mathcal{L}_{n-1+i} = \mathcal{L}_{n-1}$ for all $i \in \mathbb{N}$, i.e., we can take $s = n - 1$.

Let us state a right module analogue of Lemma 1: the right $D$-module $N = D^q/(R\,D^p)$ can be generated by $\{\tau(\Lambda_{\bullet i})\}_{i=1,\dots,q-r}$, where $\Lambda \in D^{q\times(q-r)}$ and $\Lambda_{\bullet i}$ denotes the $i^{\text{th}}$ column of $\Lambda$, iff $P \triangleq (R \quad -\Lambda)$ admits a right inverse, i.e., iff the right $D$-module $D^q/(P\,D^{(p+q-r)}) = 0$. If $R$ has full row rank, then 2 of Theorem 3 shows that the left $D$-module $E \triangleq D^{1\times(p+q-r)}/(D^{1\times q}\,P)$ is stably free iff $P$ admits a right inverse. Moreover, the right $D$-module $N$ depends only on $M$ since we can easily prove that $N$ is equal to $\operatorname{ext}^1_D(M, D)$ (up to isomorphism) (see, e.g., [29]). The right $D$-module $N$ is called the *Auslander transpose* of $M$ (see, e.g., [5]). These results are particular instances of the following result.

**Theorem 6** ([2, 3])**.** *Let $D$ be a noetherian domain, $R \in D^{q\times p}$ a full row rank matrix, i.e., $\ker_D(.R) = 0$, $0 \leq r \leq q-1$, $\Lambda \in D^{q\times(q-r)}$, $P = (R \quad -\Lambda)$ and $M = D^{1\times p}/(D^{1\times q}\,R)$ (resp., $E = D^{1\times(p+q-r)}/(D^{1\times q}\,P)$) the left $D$-module finitely presented by $R$ (resp., $P$) which defines the following* short exact sequence

$$0 \longrightarrow D^{1\times(q-r)} \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0,$$

*namely, $\alpha$ is injective, $\beta$ is surjective and $\ker\beta = \operatorname{im}\alpha$. Then, the following results are equivalent:*

1. *The left $D$-module $E$ is stably free of rank $p - r$.*

2. *The matrix* $P = (R \quad -\Lambda) \in D^{q\times(p+q-r)}$ *admits a right inverse, i.e., there exists a matrix* $S \in D^{(p+q-r)\times q}$ *such that* $P\, S = I_q$.

3. $D^q / \left(P\, D^{(p+q-r)}\right) = 0$.

4. $\{\tau(\Lambda_{\bullet i})\}_{i=1,\dots,q-r}$ *generates the right* $D$*-module* $\mathrm{ext}^1_D(M, D) = D^q/(R\, D^p)$, *where the right* $D$*-homomorphism* $\tau : D^q \longrightarrow D^q/(R\, D^p)$ *is the canonical projection and* $\Lambda_{\bullet i}$ *is the* $i^{th}$ *column of the matrix* $\Lambda$.

*Finally, the previous results depend only on the residue class* $\rho(\Lambda)$ *of* $\Lambda \in D^{q\times(q-r)}$ *in the following right* $D$*-module*

$$\mathrm{ext}^1_D\left(M, D^{1\times(q-r)}\right) \triangleq D^{q\times(q-r)} / \left(R\, D^{p\times(q-r)}\right), \tag{11}$$

*i.e., they depend only on the following row vector:*

$$(\tau(\Lambda_{\bullet 1}) \ \dots \ \tau(\Lambda_{\bullet (q-r)})) \in \mathrm{ext}^1_D(M, D)^{1\times(q-r)}.$$

A main point of the above result is that the equivalences of Theorem 6 depend only on the residue class $\rho(\Lambda)$ of $\Lambda$ in $\mathrm{ext}^1_D\left(M, D^{1\times(q-r)}\right)$, i.e., the matrix $\overline{\Lambda} \triangleq \Lambda + R\, X$ can be taken instead of $\Lambda$ for all $X \in D^{p\times(q-r)}$.

**Remark 3.** *If we take* $r = q-1$, *i.e.,* $\Lambda \in D^q$, *then Theorem 6 shows that the left* $D$*-module* $E = D^{1\times(p+1)}/(D^{1\times q}\, P)$, *where* $P = (R \quad -\Lambda) \in D^{q\times(p+1)}$, *is stably free of rank* $p-q+1$ *iff* $\tau(\Lambda)$ *generates the right* $D$*-module* $\mathrm{ext}^1_D(M, D) = D^q/(R\, D^{p+1})$, *i.e., iff* $\mathrm{ext}^1_D(M, D)$ *is a cyclic right* $D$*-module. This result was first pointed out by [30].*

**Remark 4.** If the ring $D$ admits an *involution* $\theta$, namely, a map $\theta : D \longrightarrow D$ satisfying

$$\forall\, d_1, d_2 \in D, \quad \theta(d_1 + d_2) = \theta(d_1) + \theta(d_2), \quad \theta(d_1\, d_2) = \theta(d_2)\, \theta(d_1), \quad \theta^2 = \mathrm{id}_D,$$

then $(R \quad -\Lambda)$ admits a right inverse iff $(\widetilde{R} \quad -\widetilde{\Lambda})^T$ admits a left inverse, where $\widetilde{R} \triangleq (\theta(R_{ij}))^T$ and $\widetilde{\Lambda} \triangleq (\theta(\Lambda_{ij}))^T$ (see [5]). Hence, the right $D$-module $N = D^q/(R\, D^p)$ can be generated by $\{\tau(\Lambda_{\bullet i})\}_{i=1,\dots,q-r}$ iff the left $D$-module $\widetilde{N} \triangleq D^{1\times q}/(D^{1\times p}\, \widetilde{R})$ can be generated by $\left\{\kappa\left(\widetilde{\Lambda}_{i\bullet}\right)\right\}_{i=1,\dots,q-r}$, where $\kappa : D^{1\times q} \longrightarrow \widetilde{N}$ is the canonical projection. For instance, if $D$ is a commutative ring, then $\theta = \mathrm{id}_D$ is an involution of $D$, and an involution $\theta$ of $D = A\langle \partial_1, \dots, \partial_n\rangle$ is defined by:

$$\forall\, a \in A, \quad \theta(a) = a, \quad \forall\, i = 1, \dots, n, \quad \theta(\partial_i) = -\partial_i.$$

The matrix $\widetilde{R}$ is called the *formal adjoint* of $R$ and the left $D$-module $\widetilde{N} = D^{1\times q}/(D^{1\times p}\, \widetilde{R})$ is *adjoint* of the left $D$-module $M = D^{1\times p}/(D^{1\times q}\, R)$.

Let $D = A\langle \partial\rangle$ be the ring of OD operators with coefficients in a noetherian differential integral domain $\left(A, \frac{d}{dt}\right)$, $F \in A^{n\times n}$, $G \in A^{n\times m}$, $R = \partial\, I_n - F$ and $M = D^{1\times n}/(D^{1\times n}\, R)$. Theorem 6 then shows that the right $D$-module $\mathrm{ext}^1_D(M, D) = D^n/(R\, D^n)$ is generated by $\{\tau(G_{\bullet i})\}_{i=1,\dots,m}$, where $\tau : D^n \longrightarrow D^n/(R\, D^n)$ is the canonical projection onto $\mathrm{ext}^1_D(M, D)$, iff the matrix $P \triangleq (\partial\, I_n - F \quad -G) \in D^{n\times(n+m)}$ admits a right inverse, i.e., iff the left $D$-module $E = D^{1\times(n+m)}/(D^{1\times n}\, P)$ is stably free. Using the involution $\theta$ of $D$ defined by $\theta(a) = a$ for all $a \in A$ and $\theta(\partial) = -\partial$, Remark 4 shows that $P$ admits a right inverse iff $\widetilde{P} = -(\partial\, I_n + F^T \quad G^T)^T$ admits a left inverse, i.e., iff the left $D$-module $\widetilde{E} = D^{1\times n}/(D^{1\times(n+m)}\, (-\widetilde{P}))$ defined by

$$\begin{cases} \partial\, x = -F^T\, x \\ G^T\, x = 0, \end{cases}$$

is reduced to 0. Using the results obtained in the beginning of the section, the right $D$-module $\mathrm{ext}^1_D(M, D) = D^n/(R\, D^n)$ is generated by $\{\tau(G_{\bullet i})\}_{i=1,\dots,m}$ iff the increasing sequence

$$\mathcal{L}_j = \sum_{i=0}^{j} A^{1\times m} H_i$$

of $A$-submodules of the noetherian $A$-module $A^{1\times n}$ satisfies $\mathcal{L}_s = A^{1\times n}$ for a certain $s \in \mathbb{N}$, where the matrices $H_i$'s are defined by

$$\begin{cases} H_0 = G^T, \\ H_{i+1} = H_i\, F^T - \dot{H}_i, \quad \forall\, i \in \mathbb{N}, \end{cases}$$

i.e., iff the matrix $(H_0^T \ \dots \ H_s^T)^T$ admits a left inverse, and thus iff the matrix $(G_0 \ \dots \ G_s)$ admits a right inverse, where the matrices $G_i$'s are defined by:

$$\begin{cases} G_0 = G, \\ G_{i+1} = F\, G_i - \dot{G}_i, \quad \forall\, i \in \mathbb{N}. \end{cases} \tag{12}$$

**Corollary 2.** *Let $\left(A, \frac{d}{dt}\right)$ be a noetherian differential integral domain, $F \in A^{n\times n}$, $D = A\langle\partial\rangle$ the ring of OD operators with coefficients in $A$ and $R = \partial\, I_n - F \in D^{n\times n}$. The finitely presented right $D$-module $N = D^n/(R\, D^n)$ can be generated by $m$ elements $\{\pi(G_{\bullet k})\}_{k=1,\dots,m}$, where $H \in A^{n\times m}$, iff the matrix $P = (R \quad G)$ admits a right inverse, i.e., iff there exists $s \in \mathbb{N}$ such that the matrix $(G_0 \ \dots \ G_s)$ admits a right inverse, where the matrices $G_i$'s are defined by (12).*

In control theory (see [15, 31]), a linear OD system

$$\dot{x} = F\, x + G\, u \tag{13}$$

is called *controllable* on $[t_0, t_1]$ if for every $x_1 \in \mathbb{R}^n$, there exists an essentially bounded function $u : [t_0, t_1] \longrightarrow \mathbb{R}^m$ such that $x(t_1) = x_1$, where $x$ satisfies (13) with the initial condition $x(t_0) = 0$. The sequence of matrices $G_i$'s is called the *controllability distribution* and the condition that there exists $s \in \mathbb{N}$ such that the matrix $(G_0 \ \dots \ G_s)$ admits a right inverse is the *controllability condition*.

If $\frac{d}{dt}$ is a trivial derivation of $A$, i.e., $\frac{d}{dt}\, a = \dot{a} = 0$ for all $a \in A$, then (12) yields $G_{i+1} = F^{i+1}\, G$ for all $i \in \mathbb{N}$. Using the Cayley-Hamilton theorem, the controllability condition becomes the existence of a right inverse of the following matrix:

$$\Omega \triangleq (G \quad F\, G \quad F^2\, G \quad \dots \quad F^{n-1}\, G) \in A^{n\times n\, m}. \tag{14}$$

**Example 5.** Let us consider the trivial derivation $\frac{\partial}{\partial t}$ of $A = \mathbb{Q}\langle\partial_x\rangle = \mathbb{Q}[\partial_x]$, the commutative polynomial ring $D = A\langle\partial_t\rangle = \mathbb{Q}[\partial_x, \partial_t]$ of PD operators in $\partial_t$ and $\partial_x$,

$$F = \begin{pmatrix} 0 & \partial_x + 1 \\ \partial_x^2 & 0 \end{pmatrix} \in A^{2\times 2},$$

$R = \partial_t\, I_2 - F \in D^{2\times 2}$ and the $D$-module $M = D^{1\times 2}/(D^{1\times 2}\, R)$, then $N = D^2/(R\, D^2)$ is a cyclic $D$-module iff there exists $G \in A^2$ such that the matrix $P = (R \quad -G)$ admits a right inverse by Theorem 6. Then, we get:

$$\Omega = \begin{pmatrix} G_1 & (\partial_x + 1)\, G_2 \\ G_2 & \partial_x^2\, G_1 \end{pmatrix}.$$

Since $\det \Omega = (\partial_x G_1)^2 - (\partial_x + 1)\, G_2^2$ is a polynomial in $\partial_x$ of degree at least 1, $\Omega$ is not invertible in $A$, which proves that the $D$-module $N$ is not cyclic. Now, if we consider

$$F' = \begin{pmatrix} 0 & 1 \\ \partial_x^2\,(\partial_x+1) & 0 \end{pmatrix} \in A^{2\times 2}, \tag{15}$$

$R' = \partial_t\, I_2 - F' \in D^{2\times 2}$ and $M' = D^{1\times 2}/(D^{1\times 2}\, R')$, then

$$\Omega' = \begin{pmatrix} G_1 & G_2 \\ G_2 & \partial_x^2\,(\partial_x+1)\, G_1 \end{pmatrix},$$

and thus $\det \Omega' = \partial_x^2\,(\partial_x+1)\, G_1 - G_2^2$, which shows that $N' = D^2/(R'\, D^2)$ is a cyclic $D$-module generated by $\tau((0 \quad 1)^T)$. Finally, since $N = \mathrm{ext}^1_D(M, D)$ and $N' = \mathrm{ext}^1_D(M', D)$, we obtain that $M$ and $M'$ are not isomorphic $D$-modules.

Let us give a necessary and sufficient condition for the existence of Serre's reduction.

**Theorem 7** ([2, 3])**.** *Let $D$ be a noetherian domain, $R \in D^{q\times p}$ a full row rank matrix, i.e., $\ker_D(.R) = 0$, $0 \leq r \leq q-1$ and $\Lambda \in D^{q\times(q-r)}$ such that there exists $U \in \mathrm{GL}_{p+q-r}(D)$ satisfying $(R \quad -\Lambda)\, U = (I_q \quad 0)$. If*

$$U = \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix}, \tag{16}$$

*where $S_1 \in D^{p\times q}$, $S_2 \in D^{(q-r)\times q}$, $Q_1 \in D^{p\times(p-r)}$ and $Q_2 \in D^{(q-r)\times(p-r)}$, and if we introduce the left $D$-module $L = D^{1\times(p-r)}/(D^{1\times(q-r)}\, Q_2)$ finitely presented by the full row rank matrix $Q_2$, i.e., defined by the following short exact sequence*

$$0 \longrightarrow D^{1\times(q-r)} \xrightarrow{.Q_2} D^{1\times(p-r)} \xrightarrow{\kappa} L \longrightarrow 0, \tag{17}$$

*then we have:*

$$M = D^{1\times p}/(D^{1\times q}\, R) \;\cong\; L = D^{1\times(p-r)}/(D^{1\times(q-r)}\, Q_2). \tag{18}$$

*Conversely, if $M$ is isomorphic to a left $D$-module $L$ defined by the short exact sequence (17), then there exist two matrices $\Lambda \in D^{q\times(q-r)}$ and $U \in \mathrm{GL}_{p+q-r}(D)$ satisfying:*

$$(R \quad -\Lambda)\, U = (I_q \quad 0).$$

**Corollary 3** ([2, 3])**.** *With the notations of Theorem 7, the left $D$-isomorphism (18) obtained in Theorem 7 is defined by*

$$\begin{array}{rcl} M = D^{1\times p}/(D^{1\times q}\, R) & \xrightarrow{\varphi} & L = D^{1\times(p-r)}/(D^{1\times(q-r)}\, Q_2) \\ \pi(\lambda) & \longmapsto & \kappa(\lambda\, Q_1), \end{array}$$

*and its inverse $\varphi^{-1} : L \longrightarrow M$ is defined by $\varphi^{-1}(\kappa(\mu)) = \pi(\mu\, T_1)$, where*

$$U^{-1} = \begin{pmatrix} R & -\Lambda \\ T_1 & -T_2 \end{pmatrix} \in \mathrm{GL}_{p+q-r}(D),$$

*$T_1 \in D^{(p-r)\times p}$ and $T_2 \in D^{(p-r)\times(q-r)}$. These results depend only on the residue class $\rho(\Lambda)$ of $\Lambda \in D^{q\times(q-r)}$ in the right $D$-module $\mathrm{ext}^1_D\left(M, D^{1\times(q-r)}\right)$ defined by (11).*

A straightforward consequence of Corollary 3 is the following result.

**Corollary 4** ([2, 3])**.** *Let $\mathcal{F}$ be a left $D$-module and:*

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}, \quad \ker_{\mathcal{F}}(Q_2.) = \{\zeta \in \mathcal{F}^{(p-r)} \mid Q_2\,\zeta = 0\}.$$

*Then, we have* $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(Q_2.)$ *and, more precisely:*

$$\ker_{\mathcal{F}}(R.) = Q_1\,\ker_{\mathcal{F}}(Q_2.), \quad \ker_{\mathcal{F}}(Q_2.) = T_1\,\ker_{\mathcal{F}}(R.).$$

**Corollary 5** ([2, 3])**.** *Let $R \in D^{q\times p}$ be a full row rank matrix and $\Lambda \in D^{q\times(q-r)}$ such that $P = (R \quad -\Lambda)$ admits a right inverse over $D$. Then, Theorem 7 holds when $D$ satisfies one of the following properties:*

1. *$D$ is a principal left ideal domain (e.g., the ring $A\langle\partial\rangle$ of OD operators with coefficients in a differential field $A$ such as $k$ or $k(t)$, where $k$ is a field),*

2. *$D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring over a field $k$,*

3. *$D$ is $A_n(k)$ or $B_n(k)$, where $k$ is a field of characteristic 0, and $p - r \geq 2$.*

4. *$D = A\langle\partial\rangle$ is the ring of OD operators, where $A = k[\![t]\!]$ and $k$ is a field of characteristic 0, or $k\{t\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, and $p - r \geq 2$.*

**Example 6.** Let us consider again Example 5. If the matrix $P = (\partial\, I_n - F \quad -G)$ admits a right inverse, then the left $D = A\langle\partial\rangle$-module $E = D^{1\times(n+m)}/(D^{1\times n}\,P)$ is stably free by 2 of Theorem 3.

1. If $A$ is a differential field (e.g., $A = \mathbb{R}$, $\mathbb{R}(t)$, $\mathbb{R}\{t\}[t^{-1}]$), then 1 of Corollary 5 shows that there exists a matrix $Q_2 \in D^{m\times m}$ such that:
$$M = D^{1\times n}/(D^{1\times n}\,(\partial\, I_n - F)) \cong L \triangleq D^{1\times m}/(D^{1\times m}\,Q_2). \tag{19}$$

2. If $A = k[t]$ or $k[\![t]\!]$, where $k$ is a field of characteristic 0, or $k\{t\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, and $n \geq m \geq 2$, then there exists a matrix $Q_2 \in D^{m\times m}$ such that (19) holds by 4 of Corollary 5.

3. If $\frac{d}{dt}$ is a trivial derivation of the differential ring $A = k[x_1, \ldots, x_n]$, where $k$ is a field, then 2 of Corollary 5 that there exists a matrix $Q_2 \in D^{m\times m}$ such that (19) holds. For instance, if we consider again the $D = A[\partial_t] = \mathbb{Q}[\partial_x, \partial_t]$-module $M' = D^{1\times 2}/(D^{1\times 2}\,R')$, where $R' = \partial_t\, I_2 - F'$ and $F'$ is defined by (15), then, in Example 5, we proved that the $D$-module $\operatorname{ext}^1_D(M', D) = N' = D^2/(R'\,D^2)$ was cyclic and generated by $\tau((0 \quad 1)^T)$. Using 2 of Theorem 2, we obtain that the stably free $D$-module $E = D^{1\times 3}/(D^{1\times 2}\,P)$, where $P = (R' \quad -\Lambda)$ and $\Lambda = (0 \quad 1)^T$, is free of rank 1. Using a constructive proof of the Quillen-Suslin theorem implemented in the QuillenSuslin package ([14]), we obtain:
$$\begin{pmatrix} \partial_t & -1 & 0 \\ -\partial_x^2\,(\partial_x + 1) & \partial_t & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & \partial_t \\ -\partial_t & -1 & \partial_t^2 - \partial_x\,(\partial_x^2 + 1) \end{pmatrix} = I_3. \tag{20}$$
Thus, we get $M \cong D/(D\,Q_2)$, where $Q_2 = \partial_t^2 - \partial_x\,(\partial_x^2 + 1)$. Moreover, if we consider the $D$-module $\mathcal{F} = C^\infty(\mathbb{R}^2)$ or $\mathcal{D}'(\mathbb{R}^2)$, then:
$$\ker_{\mathcal{F}}(R'.) = \begin{pmatrix} 1 \\ \partial_t \end{pmatrix} \ker_{\mathcal{F}}(Q_2.), \quad \ker_{\mathcal{F}}(Q_2.) = (1 \quad 0)\,\ker_{\mathcal{F}}(R'.).$$

**Example 7.** Let $D = A[\partial]$ be a commutative polynomial ring in $\partial$ with coefficients in a commutative ring $A$, $F \in A^{n \times n}$, $R = \partial\, I_n - F \in D^{n \times n}$ and $M = D^{1 \times n}/(D^{1 \times n}\, R)$. Let us suppose that there exists $G \in A^n$ such that the finitely presented $D$-module $E = D^{1 \times (n+1)}/(D^{1 \times n}\, P)$ is free, where $P = (\partial\, I_n - F \quad -G)$. For $i = 1, \ldots, n+1$, let $m_i \triangleq \det \widehat{P}_i$ be the determinant of the $n \times n$ submatrix of $P$ obtained by removing the $i^{\text{th}}$ column of $P$ (e.g., $m_{n+1} = \det(\partial\, I_n - F)$). Since $P$ admits a right inverse, we can prove that there exist $d_1, \ldots, d_{n+1} \in D$ such that $\sum_{i=1}^{n+1} d_i\, m_i = 1$ (see, e.g., [14]). If $T_1 = (-1)^n\, (d_1 \quad -d_2 \quad d_3 \ \ldots \ (-1)^{n+1}\, d_n)$ and $T_2 = -d_{n+1}$, then developing the determinant of the following matrix

$$\begin{pmatrix} \partial\, I_n - F & -G \\ T_1 & -T_2 \end{pmatrix} \in D^{(n+1) \times (n+1)}$$

along its last row using Laplace's formula, we get that its determinant is 1, i.e., the above unimodular matrix corresponds to the matrix $U^{-1}$ defined in Corollary 3. Moreover, we have $Q_2 = \det(\partial\, I_n - F)$ and $Q_1 = (-1)^n\, (m_1 \quad -m_2 \quad m_3 \ \ldots \ (-1)^{n+1}\, m_n)^T$. For more details, see [14]. For instance, if we consider again the end of Example 6, we get $m_1 = 1$, $m_2 = -\partial_t$ and $m_3 = \det(\partial\, I_2 - F') = \partial_t^2 - \partial_x^2\, (\partial_x + 1)$, $d_1 = 1$, $d_2 = 0$ and $d_3 = 0$, and we find again (20).

**Corollary 6** ([2, 3])**.** *With the notations of Theorem 7 and Corollary 3, if $\Lambda \in D^{q \times (q-r)}$ admits a left inverse $\Gamma \in D^{(q-r) \times q}$, i.e., $\Gamma\, \Lambda = I_{q-r}$, then $Q_1$ admits the left inverse $T_1 - T_2\, \Gamma\, R \in D^{(p-r) \times p}$ and the left $D$-module $\ker_D(.Q_1)$ is stably free of rank $r$.*

*Moreover, if the left $D$-module $\ker_D(.Q_1)$ is free of rank $r$, then there exists $Q_3 \in D^{p \times r}$ such that $W = (Q_3 \quad Q_1) \in \mathrm{GL}_p(D)$. If we write $W^{-1} = (Y_3^T \quad Y_1^T)^T$, where $Y_3 \in D^{r \times p}$ and $Y_1 \in D^{(p-r) \times p}$, then the matrix $X = (R\, Q_3 \quad \Lambda)$ is unimodular, i.e., $X \in \mathrm{GL}_q(D)$ and:*

$$V = X^{-1} = \begin{pmatrix} Y_3\, S_1 \\ Q_2\, Y_1\, S_1 - S_2 \end{pmatrix}.$$

*The matrix $R$ is then equivalent to the matrix $X \operatorname{diag}(I_r, Q_2)\, W^{-1}$, i.e.:*

$$V\, R\, W = \begin{pmatrix} I_r & 0 \\ 0 & Q_2 \end{pmatrix}.$$

*Finally, the left $D$-module $\ker_D(.Q_1)$ is free when $D$ satisfies 1 or 2 of Corollary 5 or if $D = A_n(k)$ or $B_n(k)$, where $k$ is a field of characteristic 0, and $r \geq 2$ or if $D = A\langle\partial\rangle$ is the ring of OD operators with coefficients in $A = k[\![t]\!]$, where $k$ a field of characteristic 0, or in $A = k\{t\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, and $r \geq 2$.*

**Example 8.** Let us consider a model of a two reflector antenna studied in [16, 24] which is defined by the linear differential time-delay system $\ker_{\mathcal{F}}(R.)$, where

$$R = \begin{pmatrix} \partial & -K_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial + \dfrac{K_2}{T_e} & 0 & 0 & 0 & 0 & -\dfrac{K_p}{T_e}\,\delta & -\dfrac{K_c}{T_e}\,\delta & -\dfrac{K_c}{T_e}\,\delta \\ 0 & 0 & \partial & -K_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial + \dfrac{K_2}{T_e} & 0 & 0 & -\dfrac{K_c}{T_e}\,\delta & -\dfrac{K_p}{T_e}\,\delta & -\dfrac{K_c}{T_e}\,\delta \\ 0 & 0 & 0 & 0 & \partial & -K_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial + \dfrac{K_2}{T_e} & -\dfrac{K_c}{T_e}\,\delta & -\dfrac{K_c}{T_e}\,\delta & -\dfrac{K_p}{T_e}\,\delta \end{pmatrix},$$

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\dfrac{1}{K_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{1}{K_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\dfrac{1}{K_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\dfrac{T_e + K_2}{K_1\,T_e}\,\partial & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{T_e + K_2}{K_1\,T_e}\,\partial & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\dfrac{T_e + K_2}{K_1\,T_e}\,\partial & -1 \end{pmatrix}$$

Figure 1: MatriX $S$

and $\partial\, y(t) = \dot{y}(t)$, $\delta\, y(t) = y(t-1)$ for all $y \in \mathcal{F} = C^\infty(\mathbb{R})$, and $K_1$, $K_2$, $K_c$, $K_e$, $K_p$ and $T_e$ are constant parameters. Let $D = \mathbb{Q}(K_1, K_2, K_c, K_e, T_e)\,[\partial, \delta]$ be the commutative polynomial ring of OD time-delay operators and $M = D^{1\times 9}/(D^{1\times 6}\,R)$ the $D$-module finitely presented by $R$. If we introduce the following matrix

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T \in D^{6\times 3},$$

then the matrix $S \in D^{12\times 6}$ defined in Figure 1 is a right inverse of $P = (R \quad -\Lambda) \in D^{6\times 12}$. Hence, the $D$-module $E = D^{1\times 12}/(D^{1\times 6}\,P)$ is projective, and thus free by the Quillen-Suslin theorem. Using the packages QuillenSuslin ([14]), we can compute a basis and an injective parametrization of $E$. We get that the matrix $Q \in D^{12\times 6}$ given in Figure 2 defines an injective parametrization of $E$, i.e., $\ker_D(.Q) = D^{1\times 6}\,P \cong D^{1\times 6}$. Using Theorem 7 and Corollary 4, we obtain that $M \cong L = D^{1\times 6}/(D^{1\times 3}\,Q_2)$, where $Q_2$ is the matrix defined by the last three rows of $Q$, and thus $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(Q_2.)$, i.e.:

$$\begin{cases} T_e\,\ddot{\zeta}_1(t) + K_2\,\dot{\zeta}_1(t) + (K_p + 2\,K_c)\,(K_c - K_p)\,\zeta_2(t-1) = 0, \\ T_e\,\ddot{\zeta}_3(t) + K_2\,\dot{\zeta}_3(t) + (K_p + 2\,K_c)\,(K_c - K_p)\,\zeta_4(t-1) = 0, \\ T_e\,\ddot{\zeta}_5(t) + K_2\,\dot{\zeta}_5(t) + (K_p + 2\,K_c)\,(K_c - K_p)\,\zeta_6(t-1) = 0. \end{cases}$$

We note that the equations of the previous system are uncoupled, i.e.:

$$M \cong [D^{1\times 2}/(D\,((T_e\,\partial + K_2)\,\partial \quad (K_p + 2\,K_c)\,(K_c - K_p)\,\delta)]^3. \tag{21}$$

$$Q = \left( \begin{array}{cccccc}
K_1\,T_e & 0 & 0 & 0 & 0 & 0 \\
T_e\,\partial & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & K_1\,T_e & 0 & 0 & 0 \\
0 & 0 & T_e\,\partial & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & K_1\,T_e & 0 \\
0 & 0 & 0 & 0 & T_e\,\partial & 0 \\
0 & T_e\,(K_p+K_c) & 0 & -K_c\,T_e & 0 & -K_c\,T_e \\
0 & -K_c\,T_e & 0 & T_e\,(K_p+K_c) & 0 & -K_c\,T_e \\
0 & -K_c\,T_e & 0 & -K_c\,T_e & 0 & T_e\,(K_p+K_c) \\
(T_e\,\partial+K_2)\,\partial & (K_p+2\,K_c)\,(K_c-K_p)\,\delta & 0 & 0 & 0 & 0 \\
0 & 0 & (T_e\,\partial+K_2)\,\partial & (K_p+2\,K_c)\,(K_c-K_p)\,\delta & 0 & 0 \\
0 & 0 & 0 & 0 & (T_e\,\partial+K_2)\,\partial & (2\,K_c+K_p)\,(K_c-K_p)\,\delta
\end{array} \right)$$

Figure 2: Matrix $Q$

$$W = (Q_3 \quad Q_1) =$$

$$\left(\begin{array}{ccccccccc}
0 & 0 & 0 & K_1\,T_e & 0 & 0 & 0 & 0 & 0 \\
-K_1^{-1} & 0 & 0 & T_e\,\partial & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & K_1\,T_e & 0 & 0 & 0 \\
0 & -K_1^{-1} & 0 & 0 & 0 & T_e\,\partial & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & K_1\,T_e & 0 \\
0 & 0 & -K_1^{-1} & 0 & 0 & 0 & 0 & T_e\,\partial & 0 \\
0 & 0 & 0 & 0 & T_e\,(K_p+K_c) & 0 & -K_c\,T_e & 0 & -K_c\,T_e \\
0 & 0 & 0 & 0 & -K_c\,T_e & 0 & T_e\,(K_p+K_c) & 0 & -K_c\,T_e \\
0 & 0 & 0 & 0 & -K_c\,T_e & 0 & -K_c\,T_e & 0 & T_e\,(K_p+K_c)
\end{array}\right)$$

Figure 3: Matrix $W$

We note that $\Lambda$ admits a left inverse $\Gamma$ over $D$ defined by:

$$\Gamma = \left(\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Hence, let us compute $V \in \mathrm{GL}_6(D)$ and $W \in \mathrm{GL}_9(D)$ such that $V\,R\,W = \mathrm{diag}(I_3, Q_2)$. The $D$-module $\ker_D(.Q_1)$ is a stably free and thus a free $D$-module of rank 3 by the Quillen-Suslin theorem. This last result can be checked again by computing the $D$-module $\ker_D(.Q_1)$: we have $\ker_D(.Q_1) = D^{1\times 3}\,F \cong D^{1\times 3}$, where the matrix $F$ is defined by:

$$F = \left(\begin{array}{ccccccccc} \partial & -K_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial & -K_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial & -K_1 & 0 & 0 & 0 \end{array}\right) \in D^{3\times 9}.$$

Computing a right inverse of $F$, we obtain that the matrix $Q_3 \in D^{9\times 3}$ defined by

$$Q_3 = -\frac{1}{K_1}\left(\begin{array}{ccccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right)^T$$

is such that the matrix $W$ defined in Figure 3 is unimodular, i.e., $W \in \mathrm{GL}_9(D)$. Forming the matrix $X = (R\, Q_3 \quad \Lambda) \in D^{6\times 6}$, namely,

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\dfrac{T_e\,\partial + K_2}{K_1\,T_e} & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\dfrac{T_e\,\partial + K_2}{K_1\,T_e} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{T_e\,\partial + K_2}{K_1\,T_e} & 0 & 0 & 1 \end{pmatrix},$$

then $X \in \mathrm{GL}_6(D)$. Its inverse is defined by

$$V = X^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \dfrac{T_e\,\partial + K_2}{K_1\,T_e} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{T_e\,\partial + K_2}{K_1\,T_e} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{T_e\,\partial + K_2}{K_1\,T_e} & 1 \end{pmatrix},$$

and $\overline{R} = V\,R\,W$ has finally the form $\mathrm{diag}(I_3, Q_2)$:

$$\overline{R} = V\,R\,W = \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (T_e\,\partial + K_2)\,\partial & (K_p + 2\,K_c)\,(K_c - K_p)\,\delta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \end{array}\right.$$

$$\left.\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (T_e\,\partial + K_2)\,\partial & (K_p + 2\,K_c)\,(K_c - K_p)\,\delta & 0 & 0 \\ 0 & 0 & (T_e\,\partial + K_2)\,\partial & (K_p + 2\,K_c)\,(K_c - K_p)\,\delta \end{array}\right).$$

Finally, the $D$-module $L = D^{1\times 2}/(D\,((T_e\,\partial + K_2)\,\partial \quad (K_p + 2\,K_c)\,(K_c - K_p)\,\delta)$ is clearly torsion-free and $\delta$-free ([24]) and, using (21), so is $M \cong N^3$.

**Example 9.** Let us consider again the PD linear system defined by $R' = \partial_t\, I_2 - F'$, where $F'$ is given by (15). In Example 6, we showed that we could take $\Lambda = (0 \quad 1)^T$. Since $\Lambda$

admits the left inverse $\Gamma = (0 \quad 1)$ and $D = \mathbb{Q}[\partial_x, \partial_t]$ is a commutative polynomial ring over the field $\mathbb{Q}$, Corollary 5 shows that $R'$ is equivalent to the diagonal matrix $\mathrm{diag}(1, Q_2)$, where $Q_2 = \partial_t^2 - \partial_x\,(\partial_x^2 + 1)$. Let us compute two matrices $V, W \in \mathrm{GL}_2(D)$ such that $V\,R'\,W = \mathrm{diag}(1, Q_2)$. We have $\ker_D(.Q_1) = D\,K$, where $K = (\partial_t \quad -1)$, and $K$ admits the right inverse $Q_3 = (0 \quad -1)^T$. Hence, if we introduce the following matrices

$$W = (Q_3 \quad Q_1) = \begin{pmatrix} 0 & 1 \\ -1 & \partial_t \end{pmatrix} \in \mathrm{GL}_2(D), \quad X = (R'\,Q_3 \quad \Lambda) = \begin{pmatrix} 1 & 0 \\ -\partial_t & 1 \end{pmatrix} \in \mathrm{GL}_2(D),$$

then the matrix $R'$ is equivalent to the following diagonal matrix:

$$X^{-1}\,R'\,W = \begin{pmatrix} 1 & 0 \\ 0 & \partial_t^2 - \partial_x\,(\partial_x^2 + 1) \end{pmatrix}.$$

As explained in [3], the existence of Serre's reduction can be constructively checked when $\mathrm{ext}_D^1(M, D) = D^q/(R\,D)$ is a 0-dimensional $D = k[x_1, \ldots, x_n]$-module ($k$ a computable field), namely, when $D^q/(R\,D)$ is a finite-dimensional $k$-vector space. The purpose of Section 5 is to study the corresponding case when $D = A_n(k)$ or $A\langle \partial_1, \ldots, \partial_n\rangle$, where $A = k[\![x_1, \ldots, x_n]\!]$ and $k$ a field of characteristic 0, or $A = k\{x_1, \ldots, x_n\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, i.e., the case where the right $D$-module $D^q/(R\,D)$ is holonomic. Let us give a simple example in the commutative case.

**Example 10.** Let us consider the wind tunnel model studied in [22] which is described by an OD time-delay linear system defined by the following matrix

$$R = \begin{pmatrix} \partial + a & k\,a\,\delta & 0 & 0 \\ 0 & \partial & -1 & 0 \\ 0 & \omega^2 & \partial + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix},$$

where $\partial\,y(t) = \dot{y}(t)$ is the differential operator and $\delta\,y(t) = y(t-1)$ is the time-delay operator and $\zeta$, $k$, $\omega$ and $a$ are constant parameters of the system. We note that the functional operators $\partial$ and $\delta$ commute since $(\partial \circ \delta)\,y(t) = (\dot{y})(t-1) = (\delta \circ \partial)\,y(t)$. Hence, we can consider the commutative polynomial ring $D = \mathbb{Q}(\zeta, k, \omega, a)[\partial, \delta]$ of differential time-delay operators with coefficients in the field $\mathbb{Q}(\zeta, k, \omega, a)$, where we simply denote the composition of two functional operators by the standard product (e.g., $\partial \circ \delta$ is written as $\partial\,\delta$). Now, if we introduce the finitely presented $D$-module $M = D^{1\times 4}/(D^{1\times 3}\,R)$, then $\mathrm{ext}_D^1(M, D) = D^3/(R\,D^4)$ and we can easily check by means of Gröbner basis techniques that $\mathrm{ext}_D^1(M, D)$ is a $\mathbb{Q}(\zeta, k, \omega, a)$-vector space of dimension 1 and $\tau((1 \quad 0 \quad 0)^T)$ is a basis, where $\tau : D^3 \longrightarrow \mathrm{ext}_D^1(M, D)$ is the canonical projection. If we consider the column vector $\Lambda = (1 \quad 0 \quad 0)^T$, then the matrix $P = (R \quad -\Lambda) \in D^{3\times 5}$ admits the right inverse

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -\omega^{-2}\,(\partial + 2\,\zeta\,\omega) & -\omega^2 \\ -1 & 0 & 0 \end{pmatrix},$$

which shows that $\tau((1 \quad 0 \quad 0)^T)$ generates the cyclic $D$-module $\mathrm{ext}_D^1(M, D)$. Therefore, using Remark 3, $E = D^{1\times 5}/(D^{1\times 3}\,P)$ is a stably free, and thus, a free $D$-module of rank 2 by the Quillen-Suslin theorem (see 2 of Theorem 2). Using a constructive version of the Quillen-Suslin theorem implemented in the QuillenSuslin package ([14]), we obtain that the matrix $Q \in D^{5\times 2}$ defined by

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & \omega^2 \\ 0 & \omega^2\,\partial \\ 0 & \partial^2 + 2\,\zeta\,\omega\,\partial + \omega^2 \\ \partial + a & \omega^2\,k\,a\,\delta \end{pmatrix}$$

is an injective parametrization of the free $D$-module $E$, i.e., $U = (S \quad Q) \in \mathrm{GL}_5(D)$ is such that $P\,U = (I_3 \quad 0)$. Theorem 7 then shows that $M \cong D^{1\times 2}/(D\,Q_2)$, where $Q_2 = (\partial + a \quad \omega^2\,k\,a\,\delta)$ is the last row of the matrix $Q$.

We note that $\Gamma = (1 \quad 0 \quad 0)$ is a left inverse of $\Lambda$, which shows by Corollary 6 that $\ker_D(.Q_1)$ is a free $D$-module of 1, where $Q_1$ is the matrix formed by the four first rows of $Q$. In particular, we can check that $\ker_D(.Q_1) = D^{1\times 2}\,K$, where the full row rank matrix $K$ is defined by:

$$K = \begin{pmatrix} 0 & \omega^2 & \partial + 2\,\zeta\,\omega & -\omega^2 \\ 0 & \partial & -1 & 0 \end{pmatrix}.$$

Computing a right inverse of $K$, we get that the matrix

$$Q_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ -\omega^2 & -\omega^2\,(\partial + 2\,\zeta\,\omega) \end{pmatrix}$$

is such that $W = (Q_3 \quad Q_1) \in \mathrm{GL}_4(D)$ and:

$$X = (R\,Q_3 \quad \Lambda) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{GL}_3(D), \quad V = X^{-1} = X.$$

Corollary 6 shows that the matrix $R$ is equivalent to the following block-diagonal matrix

$$V\,R\,W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \partial + a\,\omega^2 & k\,a\,\delta \end{pmatrix},$$

which proves that the wind tunnel model can be defined by an OD time-delay equation:

$$\begin{cases} \dot{x}_1(t) + a\,x_1(t) + k\,a\,x_2(t-1) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2\,x_2(t) + 2\,\zeta\,\omega\,x_3(t) - \omega^2\,u(t) = 0, \end{cases} \quad \Leftrightarrow \quad \dot{y}(t) + a\,\omega^2\,y(t) + k\,a\,v(t-1) = 0.$$

For more sophisticated examples, see [3, 9, 26].

# 5 Serre's reduction of partial differential linear systems based on holonomy

We are now in position to state the main result of this paper.

**Theorem 8.** *Let $D = A\langle \partial_1, \ldots, \partial_n \rangle$ be the ring of PD operators with coefficients in the ring $A = k[x_1, \ldots, x_n]$ or $k[\![x_1, \ldots, x_n]\!]$, where $k$ is a field of characteristic 0, or $k\{x_1, \ldots, x_n\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, $R \in D^{q\times p}$ a full row rank matrix and $M = D^{1\times p}/(D^{1\times q} R)$ the left $D$-module finitely presented by $R$. If $\mathrm{ext}^1_D(M, D) = D^q/(R\, D^p)$ is a holonomic right $D$-module, then Theorem 6 holds and we can choose a column vector $\Lambda \in D^q$ which admits a left inverse over and is such that $\tau(\Lambda)$ generates the right $D$-module $D^q/(R\, D^p)$, where $\tau : D^q \longrightarrow D^q/(R\, D^p)$ is the canonical projection onto $\mathrm{ext}^1_D(M, D)$. If $A = k[x_1, \ldots, x_n]$ and $p - q \geq 1$, then Theorem 7 and Corollaries 3 and 4 hold, i.e., $M \cong L = D^{1\times(p-q+1)}/(D\, Q_2)$, for a certain row vector $Q_2 \in D^{1\times(p-q+1)}$. Finally, if $q \geq 3$, then Corollary 6 holds, i.e., the matrix $R$ is equivalent to $\mathrm{diag}(I_{q-1}, Q_2)$.*

*Proof.* Since by hypothesis, $\mathrm{ext}^1_D(M, D)$ is a holonomic right $D$-module, Theorem 5 proves that $\mathrm{ext}^1_D(M, D)$ is cyclic and it can be generated by $\tau(\Lambda)$, where $\Lambda = (1\ d_2\ \ldots\ d_q)^T$, for certain $d_i$'s in $D$. Using Remark 3, we obtain that $E = D^{1\times(p+1)}/(D^{1\times q} P)$, where $P = (R \quad -\Lambda) \in D^{q\times(p+1)}$, is stably free of rank $p + 1 - q$. If $A = k[x_1, \ldots, x_n]$, i.e., $D = A_n(k)$, and $p + 1 - q \geq 2$, i.e., $p - q \geq 1$, then 3 of Theorem 2 shows that $E$ is a free left $D$-module of rank $p + 1 - q$, and using 3 of Theorem 3, Theorem 7 holds. Moreover, $\Gamma = (1\ 0\ \ldots\ 0)$ is a left inverse of $\Lambda$, and thus Corollary 6 holds. Finally, if $r = q - 1 \geq 2$, i.e., $q \geq 3$, then the stably free left $D$-module $\ker_D(.Q_1)$ of rank $r$ is free by Stafford's theorem (see 3 of Theorem 2) and Corollary 6 proves that $R$ is equivalent to $\mathrm{diag}(I_{q-1}, Q_2)$ for a certain row vector $Q_2 \in D^{1\times(p-q+1)}$. □

In the case of $D = A_n(k)$, where $k$ is a field of characteristic 0, we can use [19] and [27] to obtain the following algorithm implemented in the Serre package ([9]).

**Algorithm 1.**

- **Input:** A full row rank matrix $R \in D^{q\times p}$ such that $p - q \geq 1$ and the right $D = A_n(k)$-module $N \triangleq D^q/(R\, D^p)$ is holonomic ($k$ a field of characteristic 0).
- **Output:** A matrix $Q_2 \in D^{1\times(p-q+1)}$ such that $M \cong D^{1\times(p-q+1)}/(D\, Q_2)$. Moreover, if $q \geq 3$, then two more matrices $V \in \mathrm{GL}_q(D)$ and $W \in \mathrm{GL}_p(D)$ are returned such that:

$$V\, R\, W = \mathrm{diag}(I_{q-1}, Q_2).$$

1. Use Algorithm 3 of [19] to compute a column vector $\Lambda = (1 \quad d_2\ \ldots\ d_q)^T$ such that $N \triangleq D^q/(R\, D^p) = D\, \tau(\Lambda)$, where $\tau : D^q \longrightarrow N$ is the canonical projection onto $N$, i.e. such that $N = (P\, D^{p+1})/(R\, D^p)$, where $P = (R \quad -\Lambda) \in D^{q\times(p+1)}$.

2. Using Algorithm 3 of [27], compute two matrices

$$\begin{cases} Q = (Q_1^T \quad Q_2^T)^T \in D^{(p+1)\times(p-q+1)}, \\ T = (T_1 \quad T_2) \in D^{(p-q+1)\times(p+1)} \end{cases}$$

   where $Q_1 \in D^{p\times(p-q+1)}$, $Q_2 \in D^{1\times(p-q+1)}$, $T_1 \in D^{(p-q+1)\times p}$ and $T_2 \in D^{(p-q+1)\times 1}$, such that $\ker_D(.Q) = D^{1\times q} P$ and $T\, Q = I_{p-q+1}$.

3. If $q \leq 2$, then return the matrix $Q_2$.

4. Else, compute a matrix $K \in D^{r\times p}$ such that $\ker_D(.Q_1) = D^{1\times r}\, K$.
5. Compute a matrix $L \in D^{s\times r}$ such that $\ker_D(.K) = D^{1\times s}\, L$.
6. If $L = 0$, i.e., $\ker_D(.K) = 0$, then $r = q-1$.
   (a) Compute a right inverse $Q_3 \in D^{p\times(q-1)}$ of the matrix $K \in D^{(q-1)\times p}$.
   (b) Form the matrices $X = (R\,Q_3 \quad \Lambda) \in \mathrm{GL}_q(D)$ and $W = (Q_3 \quad Q_1) \in \mathrm{GL}_p(D)$.
   (c) Compute $V = X^{-1}$.
   (d) Return the matrices $Q_2$, $V$ and $W$.
7. Else, i.e., $L \neq 0$, then:
   (a) Using Algorithm 4 of [27], compute $F \in D^{r\times(q-1)}$ and $G \in D^{(q-1)\times r}$ such that $\ker_D(.F) = D^{1\times s}\, L$ and $G\,F = I_{q-1}$.
   (b) Form the full row rank matrix $G\,K \in D^{(q-1)\times p}$.
   (c) Compute a right inverse $Q_3 \in D^{p\times(q-1)}$ of the matrix $G\,K \in D^{(q-1)\times q}$.
   (d) Form the matrices $X = (R\,Q_3 \quad \Lambda) \in \mathrm{GL}_q(D)$ and $W = (Q_3 \quad Q_1) \in \mathrm{GL}_p(D)$.
   (e) Compute $V = X^{-1}$.
   (f) Return the matrices $Q_2$, $V$ and $W$.

**Example 11.** Let us consider the commutative polynomial ring $D = \mathbb{Q}\,[\partial_x, \partial_y]$ of PD operators and the $D$-module $M = D^{1\times 3}/(D^{1\times 2}\, R)$ finitely presented by $R$ defined by:

$$R = \begin{pmatrix} \partial_x & \partial_y & 0 \\ 0 & \partial_x & \partial_y \end{pmatrix} \in D^{2\times 3}. \tag{22}$$

The matrix $R$ defines the equation $R\,\sigma = 0$ of the equilibrium of the *stress tensor* in $\mathbb{R}^2$:

$$\begin{cases} \partial_x\,\sigma^{11} + \partial_y\,\sigma^{12} = 0, \\ \partial_x\,\sigma^{12} + \partial_y\,\sigma^{22} = 0. \end{cases} \tag{23}$$

We can check that $\mathrm{ext}^1_D(M, D) = D^2/\left(R\,D^3\right)$ is a $\mathbb{Q}$-vector space of dimension 3 and a basis of $\mathrm{ext}^1_D(M, D)$ is defined by $\tau((1 \quad 0)^T)$, $\tau((0 \quad 1)^T)$ and $\tau((0 \quad \partial_x)^T)$, where $\tau : D^2 \longrightarrow D^2/(R\,D^3)$ is the canonical projection. Hence, without loss of generality, we can assume that $\Lambda$ has the form $\Lambda = (a \quad b + c\,\partial_x)^T$, where $a$, $b$ and $c$ are three arbitrary constants. Considering the new ring $D' = \mathbb{Q}[a, b, c]\,[\partial_x, \partial_y]$, $P = (R \quad -\Lambda)$ and the $D'$-module $E = D'^{1\times 4}/(D'^{1\times 2}\, P)$, then, using Gröbner basis techniques, we can check that the matrix $P$ does not admit a right inverse with entries in $D'$. According to Theorem 3, we obtain that the $A$-module $E$ is not a stably free $D'$-module, which proves that (23) cannot be defined by a sole PD equation with constant coefficients, and the minimal number of generators $\mu(M)$ of the $D$-module $M$ is 3.

Let $M' = B^{1\times 3}/(B^{1\times 2}\, R)$ be the left $B = A_2(\mathbb{Q})$-module finitely presented by $R$. The right $B$-module $\mathrm{ext}^1_B(M', B) = B^2/(R\,B^3)$ is holonomic and thus cyclic by Proposition 5. The element $\tau(\Lambda)$ of $\mathrm{ext}^1_B(M', B)$, where $\Lambda = (1 \quad x)^T$, generates $\mathrm{ext}^1_B(M', B)$ since the matrix $P = (R \quad -\Lambda) \in B^{2\times 4}$ admits the following right inverse:

$$T = \begin{pmatrix} -x & 1 \\ -x^2 & x \\ -x^3 & x^2 \\ -x\,(x\,\partial_y + \partial_x) - 2 & \partial_x + x\,\partial_y \end{pmatrix}.$$

The left $B$-module $E' = B^{1\times 4}/(B^{1\times 2}\, P)$ is then stably free of rank 2 (see Remark 3), i.e., free by Stafford's theorem (see 3 of Theorem 2). Using the STAFFORD package ([27]), an injective parametrization of $E'$ is defined by

$$Q = \begin{pmatrix} \partial_y & \partial_x \\ x\,\partial_y & x\,\partial_x - 1 \\ x^2\,\partial_y - 1 & x\,\partial_x - x \\ (\partial_x + x\,\partial_y)\,\partial_y & (\partial_x + x\,\partial_y)\,\partial_x - \partial_y \end{pmatrix},$$

which yields $M' \cong B^{1\times 2}/(B\,((\partial_x + x\,\partial_y)\,\partial_y \quad (\partial_x + x\,\partial_y)\,\partial_x - \partial_y))$.

Since $\Gamma = (1 \quad 0)$ is a left inverse of $\Lambda$, using Corollary 6, we obtain the following unimodular matrices:

$$W = \begin{pmatrix} -1 & \partial_y & \partial_x \\ -x & x\,\partial_y & x\,\partial_x - 1 \\ -x^2 & x^2\,\partial_y - 1 & x\,(x\,\partial_x - 1) \end{pmatrix}, \quad W^{-1} = \begin{pmatrix} x\,\partial_x & x\,\partial_y - \partial_x & -\partial_y \\ 0 & x & -1 \\ x & -1 & 0 \end{pmatrix},$$

$$X = \begin{pmatrix} -(\partial_x + x\,\partial_y) & 1 \\ -x\,(\partial_x + x\,\partial_y) - 1 & x \end{pmatrix}, \quad V = X^{-1} = \begin{pmatrix} x & -1 \\ x^2\,\partial_y + x\,\partial_x + 2 & -(\partial_x + x\,\partial_y) \end{pmatrix}.$$

Then, the matrix $R$ defined by (22) is equivalent to the following block-diagonal matrix

$$V\,R\,W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\partial_x + x\,\partial_y)\,\partial_y & (\partial_x + x\,\partial_y)\,\partial_x - \partial_y \end{pmatrix},$$

which proves that (23) is equivalent to the following PD equation

$$(\partial_x + x\,\partial_y)\,\partial_y\,\tau_2 + (\partial_x + x\,\partial_y)\,\partial_x\,\tau_3 - \partial_y\,\tau_3 = 0,$$

under the following invertible transformations:

$$\begin{cases} \sigma^{11} = \partial_y\,\tau_2 + \partial_x\,\tau_3, \\ \sigma^{12} = x\,\partial_y\,\tau_2 + x\,\partial_x\,\tau_3 - \tau_3, \\ \sigma^{22} = x^2\,\partial_y\,\tau_2 - \tau_2 + x^2\,\partial_x\,\tau_3 - x\,\tau_3, \end{cases} \qquad \begin{cases} \tau_1 = x\,(\partial_x\,\sigma^{11} + \partial_y\,\sigma^{12}) - (\partial_x\,\sigma^{12} + \partial_y\,\sigma^{22}) = 0, \\ \tau_2 = x\,\sigma^{12} - \sigma^{22}, \\ \tau_3 = x\,\sigma^{11} - \sigma^{12}. \end{cases}$$

We note that we have lost the symmetry of (23). It would be interesting to get a more symmetric equivalent PD equation by considering another cyclic vector of $\mathrm{ext}^1_E(M', E)$.

If $D$ is a noetherian domain and $M = D^{1\times p}/(D^{1\times q}\, R)$ a left $D$-module finitely presented by a full row rank matrix $R$, then we can prove that the right $D$-module $\mathrm{ext}^1_D(M, D) = D^q/(R\, D^p)$ is torsion ([5]). Then, using Theorem 4 and 4 of Corollary 5, we obtain the following corollary of Theorem 8.

**Corollary 7.** *Let $D = A\langle\partial\rangle$ be the ring of OD operators with coefficients in $A = k[t]$ or $k[[t]]$ and $k$ is a field of characteristic 0, or $A = k\{t\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, $R \in D^{q\times p}$ a full row rank matrix and $M = D^{1\times p}/(D^{1\times q}\, R)$ the left $D$-module finitely presented by $R$. Then, Theorem 6 holds and $\Lambda \in D^q$ can be chosen so that it admits a left inverse over $D$ and $\tau(\Lambda)$ generates the right $D$-module $\mathrm{ext}^1_D(M, D) = D^q/(R\, D^p)$. Moreover, if $p - q \geq 1$, then Theorem 7 and Corollaries 3 and 4 hold. Finally, if $q \geq 3$, then Corollary 6 holds.*

Corollary 7 shows that every analytic linear OD system defined by a full row rank matrix with at least one column more than rows is isomorphic to an analytic linear OD equation. Moreover, if the system has at least 3 equations, then the system is equivalent to a sole OD equation. These results are particularly meaningful in control theory. For instance, if we consider the analytic linear OD system

$$\dot{x} = F\,x + G\,u, \tag{24}$$

where $F \in A^{n\times n}$ and $G \in A^{n\times m}$, where $A = k[t]$ or $k[\![t]\!]$ and $k$ is a field of characteristic 0, or $A = k\{t\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, then we get $M = D^{1\times(n+m)}/(D^{1\times n}\,R)$, where $D = A\langle\partial\rangle$ and $R = (\partial\, I_n - F \quad -G) \in D^{n\times(n+m)}$. In particular, we have $p = n+m$, $q = n$ and $\mathrm{rank}_D(M) = p - q = m$. Therefore, if $m \geq 1$, i.e., if the dimension of the input vector $u$ of (24) is at least 1, i.e., $F \neq 0$, then $M \cong D^{1\times(m+1)}/(D\,Q_2)$ for a certain row vector $Q_2 \in D^{1\times(m+1)}$. Moreover, if $n \geq 3$, i.e., if the dimension of the state vector $x$ of (24) is at least 3, then (24) is equivalent to the linear analytic OD equation $Q_2\,\zeta = 0$.

Since the rings $D = B_1(k)$, $k[\![t]\!][t^{-1}]\langle\partial\rangle$, where $k$ is a field of characteristic 0, or $k\{t\}[t^{-1}]\langle\partial\rangle$, where $k = \mathbb{R}$ or $\mathbb{C}$, are simple principal left ideal domains (see, e.g., [1, 20]), using the concept of *Jacobson normal form*, namely, a generalization of the Smith normal form to principal left or right ideal domains (see, e.g., [12, 18, 32]), for every matrix $R \in D^{q\times p}$, there exist $V \in \mathrm{GL}_q(D)$, $W \in \mathrm{GL}_p(D)$ and $d \in D$ such that

$$V\,R\,W = \mathrm{diag}(1, \ldots, 1, d, 0, \ldots, 0),$$

i.e., $R$ is equivalent to the diagonal matrix $\overline{R} = \mathrm{diag}(1, \ldots, 1, d, 0, \ldots, 0)$, for a certain $d \in D$. In particular, if $R$ has full row rank, then $R$ is equivalent to $\mathrm{diag}(1, \ldots, 1, d)$.

Now, if $D = A_1(k)$, $k[\![t]\!]\langle\partial\rangle$, where $k$ is a field of characteristic 0, or $k\{t\}\langle\partial\rangle$, where $k = \mathbb{R}$ or $\mathbb{C}$, and $R \in D^{q\times p}$, then the Jacobson normal form of $R$ can be computed by considering the injection of $D$ into the simple principal left ideal domain $D'$, where $D'$ is respectively $B_1(k)$, $k[\![t]\!][t^{-1}]\langle\partial\rangle$ and $k\{t\}[t^{-1}]\langle\partial\rangle$. Therefore, there exist $V \in \mathrm{GL}_q(D')$, $W \in \mathrm{GL}_p(D')$ and $e \in D'$ such that $V\,R\,W = \mathrm{diag}(1, \ldots, 1, e, 0, \ldots, 0)$. However, singularities may have been introduced in $e$, $V$ and $W$. Corollary 7 shows that there always exist three matrices $Q_2 \in D^{1\times(p-q+1)}$, $X \in \mathrm{GL}_q(D)$ and $Y \in \mathrm{GL}_p(D)$ such that $X\,R\,Y = \mathrm{diag}(I_{q-1}, Q_2)$. Since the entries of $Q_2$, $X$, $Y$, $X^{-1}$ and $Y^{-1}$ belong to $D$, no singularity can appear.

**Example 12.** Let $M = D^{1\times 4}/(D^{1\times 3}\,R)$ be the left $D = A_1(\mathbb{Q})$-module finitely presented by the following matrix:

$$R = \begin{pmatrix} t\,\partial + 2 & 0 & \partial & \partial \\ \partial + t^2 & 0 & \partial^2 + 1 & t \\ t^2\,(\partial^2 - 1) + 2\,t\,\partial & -t\,\partial & t\,\partial & t\,\partial^2 - t \end{pmatrix} \in D^{3\times 4}. \tag{25}$$

Using Algorithm 3 of [19], we obtain that the column vector $\Lambda = (0 \quad 0 \quad 1)^T$ is such that the matrix $P = (R \quad -\Lambda)$ admits the following right inverse:

$$S = \begin{pmatrix} \partial^2 + 1 & -\partial & -\partial & -t\,(\partial^2 + 1) & 0 \\ -\partial & 1 & 1 & t\,\partial & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}^T.$$

In other words, the right $D$-module $\mathrm{ext}^1_D(M, D) = D^3/(R\,D^4)$ is cyclic and is generated by $\tau(\Lambda)$, and thus the left $D$-module $E = D^{1\times 5}/(D^{1\times 3}\,P)$ is stably free of rank 2, i.e., is free of rank 2 by Stafford's theorem (see 3 of Theorem 2). Computing an injective parametrization of $E$ by means of the STAFFORD package, we obtain that the matrix $Q = (Q_1^T \quad Q_2^T)^T \in D^{5\times 2}$, where

$$Q_1 = \begin{pmatrix} \partial^3 + (1-t)\,\partial - 1 & 0 \\ -\partial^2 - \partial + t & 1 \\ -\partial^2 + t & 0 \\ -t\,\partial^3 + t\,(t-1)\,\partial + t - 1 & 0 \end{pmatrix}, \quad Q_2 = (t \quad -t\,\partial),$$

satisfies $\ker_D(.Q) = D^{1\times 3}\,P$ and $T\,Q = I_2$, where:

$$T = \begin{pmatrix} -t & 0 & 0 & -1 & 0 \\ 1 & 1 & \partial - 1 & 0 & 0 \end{pmatrix}.$$

Thus, we get $M \cong D^{1\times 2}/(D\,Q_2)$. Moreover, since $\Lambda$ admits the left inverse $\Gamma = (0 \quad 0 \quad 1)$, $R$ is equivalent to $\operatorname{diag}(I_2, Q_2)$. More precisely, we have $\ker_D(.Q_1) = D^{1\times 2}\,K$, where

$$K = \begin{pmatrix} t\,\partial + 2 & 0 & \partial & \partial \\ \partial + t^2 & 0 & \partial^2 + 1 & t \end{pmatrix},$$

and the right inverse $Q_3$ of $K$, defined by

$$Q_3 = \begin{pmatrix} \partial^2 + 1 & 0 & -\partial & -t\,(\partial^2 + 1) \\ -\partial & 0 & 1 & t\,\partial \end{pmatrix}^T,$$

is such that $W = (Q_3 \quad Q_1) \in \mathrm{GL}_4(D)$. Then, we have:

$$X = (R\,Q_3 \quad \Lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -t\,\partial^2 & t\,\partial & 1 \end{pmatrix}, \quad V = X^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t\,\partial^2 & -t\,\partial & 1 \end{pmatrix}.$$

We obtain that the matrix $R$ is equivalent to the following block-diagonal matrix:

$$V\,R\,W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t & -t\,\partial \end{pmatrix}.$$

If we compute a Jacobson normal form $J$ of the matrix $R$, then we get

$$J = Y\,R\,Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

for certain matrices $Y$ and $Z$ containing large entries in $B_1(\mathbb{Q})$, i.e., $Y \in \mathrm{GL}_3(B_1(\mathbb{Q}))$ and $Z \in \mathrm{GL}_4(B_1(\mathbb{Q}))$. Finally, we note that the left $D$-module $L = D^{1\times 2}/(D\,Q_2)$ admits the non-trivial torsion element $z = z_1 - \partial\,z_2$, where $z_1 = \kappa((1 \quad 0))$ and $z_2 = \kappa((0 \quad 1))$ are the generators of $L$ satisfying $t\,(z_1 - \partial\,z_2) = 0$, whereas the left $D' \triangleq B_1(\mathbb{Q})$-module $D'^{1\times 2}/(D'\,Q_2)$ is free of rank 1.

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Centre de recherche INRIA Saclay – Île-de-France
Parc Orsay Université - ZAC des Vignes
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